The divergence of Banach space valued random variables on Wiener space

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Abstract

The domain of definition of the divergence operator δ on an abstract Wiener space (W, H, μ) is extended to include $W-$valued and $W \otimes W-$ valued "integrands". The main properties and characterizations of this extension are derived and it is shown that in some sense the added elements in δ's extended domain have divergence zero. These results are then applied to the analysis of quasiinvariant flows induced by W-valued vector fields and, among other results, it turns out that these divergence-free vector fields "are responsible" for generating measure preserving flows.

Key Words: Abstract Wiener Space, Divergence, Flows.

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1 Introduction

The classical Malliavin calculus is based on the notions of gradient and divergence operators in a Hilbert space setting. The gradient operation deals traditionally with the directional derivative of real or Hilbert space valued random variables in the direction of elements of the Cameron-Martin space and the divergence operator is introduced by duality with respect to the gradient operator. This setup, centered around separable Hilbert spaces, has proved to be a powerful tool in solving many problems. However, it needed to be extended in several cases, whether in stochastic analysis on manifolds ([6], [7], [4]), in the analysis on abstract Wiener spaces, ([8], [14], [18], [16]) or in considerations associated with extending the Malliavin calculus to include measure preserving transformations of the Wiener path ([9], [22]).

Let $\{W, H, \mu\}$ be an abstract Wiener space (cf. next section), let $\{e_i\}_{i\in\mathbb{N}}$ be a smooth ONB in H and $\{\eta_i\}_{i\in\mathbb{N}}$ a sequence of i.i.d. $N(0,1)$ random variables on the Wiener space. Then, by the Ito-Nisio theorem ([12]) $Y_n = \sum_1^n \eta_i e_i$ converges in W and the limit, say Y, is a measure preserving transformation on $\{W, \mu\}$. There is no reason to expect that in general the difference $Y(\omega) - \omega$ will be H valued (indeed, $Y(\omega) = -\omega$ is one such counterexample). Moreover, for a collection $Y_t = \sum_1^\infty \eta_i(t,\omega) e_i$ of such measure preserving transformations, $(dY_t/dt)_{t=0}$ even if it exists, need not be H-valued. Consequently, the analysis of measure invariance (and related) flows on Wiener space requires the study of W-valued, rather than only Cameron-Martin, vector fields ([2], [9], [5], [22]).

In this paper we (a) extend the domain of definition of the divergence operator to include Banach space valued random variables and derive the main properties and representation of this extension and (b) apply the results of the first part to the analysis of flows on Wiener space.

In the next section we first summarize the background and notation for later reference. Differentiation of random variables is generalized by stipulating differentiability subspaces other than H, smaller or larger, yielding Sobolev spaces which respectively contain or are contained in the standard ones $\mathbb{D}_{p,1}$.

In Section 3 we extend the domain of definition of the divergence from H-valued to appropriate W-valued, and even W^{**}–valued, random variables. The main properties of the extended divergence are derived and Shigekawa's decomposition [17] of the domain of this divergence into *exact* and *divergence-free* subspaces of "integrands" is shown to hold in this generalized setup as well. In fact it turns out that it is the class of divergence-free

integrands which is extended but not the class of exact integrands.

The classical divergence also operates on $H^{\otimes 2}$, the Hilbert-Schmidt operators on H. Section 3 also contains its extension in this case to random operators from W^* to an arbitrary Banach space Y. This construction is then applied in Section 4 for $Y = W^{**}$ (in this case operators from W^* to W^{**} can be seen as bilinear forms on W^*) to derive the representation of any divergence-free integrand as the divergence of a random antisymmetric bilinear form on W^*. In [17], Shigekawa constructed a general setup for H-valued differential forms on Wiener space and derived the H Hodge-Kodaira theory for this setup. Our results constitute an extension of this theory beyond H, restricted to forms of order 1 (W^{**}–valued random variables) and of order 2 (random bilinear forms on W^*, not necessarily antisymmetric).

Section 5 starts with an introduction to measure preserving transformation of Wiener space. The results of Sections 3 and 4 are applied in Section 5 to derive new results concerning flows generated by W–valued vector fields, extending the results of [9] and [22] on measure preserving flows to general flows.

Section 6 deals with (a) The notion of adapted W-valued vector fields and conditions under which the flows they generate are adapted and (b) The relation between the flow equation of Section 5 and a class of scalar valued partial differential equations motivated by the non-random case introduced by P-L. Lions in [11].

In some of the results presented in this paper it is required that a W-valued random variable, say $u(\omega)$, or a collection of such r.v.'s, have the "representability" property that for some orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H whose elements are in W^*

$$\left\| u - \sum_{i=1}^{n} {}_{W}\langle u, e_i \rangle_{W^*} e_i \right\|_W \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{a.s.} \quad \text{or in} \quad L^p(\mu) \tag{1.1}$$

(the dual use of e_i both as an element of W and of W^* will be further clarified later). The representation (1.1) of u will obviously hold if $\{e_i\}_{i\in\mathbb{N}}$ is a Schauder basis of W, but it might still be valid even if W does not possess such a basis. Note that if u is representable in this sense, and if $T : W \to W$ is measurable with $T^*\mu \ll \mu$, then so is $u \circ T$. Finding appropriate conditions under which a W-valued r.v u is representable as in (1.1) seems to be delicate.

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2 Preliminaries

In this section we first recall some notions of stochastic analysis in abstract Wiener space, as well as the Ornstein–Uhlenbeck semigroup and its generator, the number operator. In the second part we generalize the notion of subspaces of differentiability to other than H.

2.1 Notation and Generalities

The basic object in this paper will be an (infinite dimensional) abstract Wiener space (W, H, μ), based on a separable Banach space W with a densely embedded Hilbert space H, and a Gaussian measure μ on W under which each $l \in W^*$ becomes an $N(0, |h|_H^2)$ random variable. The embeddings $i : H \to W$ and $i^* : W^* \to H$ will not always be written explicitly; thus, for example, an element $e \in W^*$ will also be considered to be an element in H or in W, the distinction being clear from the context, as for example in (1.1).

In H, the inner product is denoted by $(\cdot,\cdot)_H$ and the notation $|\cdot|_H$ for the norm in H has already been used in the previous paragraph. An orthonormal basis (ONB) $\mathcal{E} = \{e_i\}_{i=1}^{\infty}$ of H will be said to be **smooth** if $e_i \in W^*$ for all i. The norms in W and W^* are $\|\cdot\|_W$ and $\|\cdot\|_{W^*}$ respectively, while the natural pairing between $l \in W^*$ and $w \in W$ (resp. between $w^{**} \in W^{**}$ and $l \in W^*$) is denoted ${}_W\langle w, l\rangle_{W^*}$ (resp. ${}_{W^*}\langle l, w^{**}\rangle_{W^{**}}$). Any of these subscripts may be omitted if no confusion arises.

We recall the canonical zero–mean Gaussian field $\{\delta h,\ h \in H\}$ whose correlation is given by H's inner product. In particular, $\delta l(\omega) = {}_W\langle \omega, l\rangle_{W^*}$ a.s for every $l \in W^*$. For $1 \le p \le \infty$, $L^p(\mu)$ or $L^p(W, \mu)$ will denote $L^p(W, \mathcal{F}, \mu)$ where $\mathcal{F} := \sigma(\delta h, h \in H)$, the sigma–algebra generated by the canonical Gaussian field. The same applies to $L^p(\mu; Y)$ for any other Banach space Y.

The space of bounded linear operators from a Banach space X to a Banach space Y is denoted $L(X, Y)$ equipped with the operator norm $\|A\|_{L(X,Y)} = \sup\{\|Ax\|_Y,\ \|x\|_X \le 1\}$ and $L(X) := L(X, X)$. The space of bilinear forms on a Banach space X is denoted $M_2(X)$ and is equipped with the norm $\|T\|_{M_2(X)} = \sup\{|T(x, x')|,\ x, x' \in X,\ \|x\|_X = \|x'\|_X = 1\}$.

The reader is assumed to be familiar with the basic notions of the Malliavin calculus, i.e., the gradient ∇ and the divergence δ applied to the Sobolev spaces $\mathbb{D}_{p,k}$ ($\nabla : \mathbb{D}_{p,k} \to \mathbb{D}_{p,k-1}(H)$ and $\delta : \mathbb{D}_{p,k}(H) \to \mathbb{D}_{p,k-1}$). We will however be somewhat more explicit about the Ornstein-Uhlenbeck semigroup and conclude this subsection with a summary of some

of its associated facts as needed in later sections (cf., e.g., [18], [19], [15]).

Let $(\widetilde{W}, H, \mu)$ be an independent copy of (W, H, μ) and $f \in L^p(W, \mu)$. The Ornstein-Uhlenbeck semigroup on $L^p(\mu)$, $p \geq 1$, is defined by the Mehler formula

$$T_t f(\omega) = E_W f\left(e^{-t}\omega + \sqrt{1 - e^{-2t}}\,\widetilde{\omega}\right) \tag{2.1}$$

where E_W denotes the conditional expectation conditioned on W. The family $\{T_t\}_{t\geq 0}$ is a contraction, self-adjoint semigroup, whose infinitesimal generator $-\mathcal{L}$ satisfies

$$(1 + \mathcal{L})^{-\beta} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\beta - 1} e^{-t} T_t dt\,. \tag{2.2}$$

Consequently $(1 + \mathcal{L})^{-\beta}$, $\beta > 0$, is a bounded operator on $L_p(\mu)$ for every $\beta > 0$. Moreover,

(i) Since T_t is self-adjoint, so are $(1+\mathcal{L})^{-\beta}$, $\beta > 0$, and $\mathcal{L}$;

(ii) $\mathcal{L} = \delta \circ \nabla$ on $\mathbb{D}_{p,2}$;

(iii) $\nabla(1 + \mathcal{L})^{-\frac{1}{2}}$ is a bounded linear operator from $L^p(\mu)$ to $L^p(\mu; H)$ for any $p \in (1, \infty)$;

(iv) $(1+\mathcal{L})^{\beta}\nabla f = \nabla \mathcal{L}^{\beta} f$ for all real β and every $f \in \mathbb{D}_{p,1}$ with $Ef = 0$.

The definition of T_t can be extended to f's taking values in a separable Banach space Y for which $E\|f\|_Y^p < \infty$ (i.e. to $L^p(\mu; Y)$) in which case the expectation in (2.1) is defined as a Bochner integral. Formula (2.2) and the boundedness of $(1 + \mathcal{L})^{-\beta}$, $\beta > 0$, remain true. The Ornstein-Uhlenbeck semigroup T_t for Y-valued and real valued functions are related via

$$T_t\, {}_Y\langle f(\omega), e\rangle_{Y^*} = {}_Y\langle T_t f(\omega), e\rangle_{Y^*} \qquad \text{for all } f \in L^p(\mu; Y),\ e \in Y^*, \tag{2.3}$$

In particular, if $a(\omega)$ is representable in the sense of (1.1), then

$$T_t a = \sum_i T_t (a, e_i)_Q\, e_i$$

and similarly for $\mathcal{L}$, etc. Moreover, (i)—(iv) ,under obvious modifications, remain true.

2.2 Stochastic Differentiation

Let

$$\mathcal{S}_n = \left\{ \Phi = \varphi(\delta l_1, \ldots, \delta l_n) \;\middle|\; l_i \in W^*,\; i = 1, \ldots, n, \quad \varphi \in C_b^\infty(\mathbb{R}^n) \right\}. \tag{2.4}$$

For any $\Phi \in \mathcal{S}_n$ represented as in (2.4), its gradient is the W^*–valued random variable

$$\nabla\Phi = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}(\delta l_1, \ldots, \delta l_n)\, l_i, \tag{2.5}$$

and this definition can be easily seen not to depend on Φ's particular representation. Denote

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n. \tag{2.6}$$

The classical Sobolev completion of $\mathcal{S}$ yields a space of functionals differentiable along H. In fact, other Sobolev spaces can be obtained by considering different subspaces of differentiability. Given a Banach space $(Z, \| \|_Z)$ continuously embedded in W (the elements of Z will be the directions of differentiability; cf. Remark 2.1 below)$W^* \subset Z^*$ and $\nabla\Phi$ can be viewed as being Z^* valued; indeed

$$_{Z}\langle z, \nabla\Phi(\omega)\rangle_{Z^*} = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}(\delta l_1, \ldots, \delta l_n)\; {}_{Z}\langle z, l_i\rangle_{Z^*}. \tag{2.7}$$

Thus for all $\Phi \in \mathcal{S}$ and $p \in [1, \infty)$ consider the Sobolev norms on S

$$\|\Phi\|_{p,1;Z} \quad = \quad \left(\|\Phi\|^p_{L^p(\mu)} + \|\nabla\Phi\|^p_{L^p(\mu;Z^*)} \right)^{\frac{1}{p}} \tag{2.8}$$

and denote $\mathcal{S}$'s completion according to this norm by $\mathbb{D}^Z_{p,1}$.

Example: The Hermite polynomials $H_n(x) = \frac{(-1)^n}{\sqrt{n!}}\, e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}} \right)$, $n = 0, 1, 2, \ldots$ satisfy $EH_n(X)H_m(X) = \delta_{n,m}$ for $X \sim N(0,1)$ and $H_n' = \sqrt{n}H_{n-1}$ for $n = 1, 2, \ldots$. Given an ONB e_n of H, and by Levy's criterion, the sequence of random variables $a_m = \sum_{n=1}^{m} \frac{H_{2n}(\delta e_n)}{\sqrt{n}\log n} \in \mathcal{P}_n$ converges in L^2 and a.s. to, say, a and $\nabla a_m = \sum_{n=1}^{m} \frac{\sqrt{2}}{\log n} H_{2n-1}(\delta e_n) e_n$. This H–valued sequence does not converge in $L^2(\mu; H)$; If, however, for the case where the W space is the completion of H under the norm $\|u\|_W = \left(\sum_i \left| \frac{1}{n} \langle u, e_i\rangle \right|^2 \right)^{\frac{1}{2}}$ or $\|u\|_W =$

$\|Qu\|_H$ where Q is a Hilbert–Schmidt operator on H, ∇a_m converges in W. Therefore, in this case $a \in \mathbb{D}_{2,1}^{W^*}$ but $a \notin \mathbb{D}_{2,1}^{H}$. More generally, for any abstract Wiener space, W, we can embed a W_0 of the form defined above, i.e. $H \subset W_0 \subset W$, and then $a \in \mathbb{D}_{2,1}^{W_0^*}$.

Remark 2.1 In view of (2.7) it is natural to think of $\nabla\Phi$ being characterized by

$$ {}_Z\langle z, \nabla\Phi(\omega)\rangle_{Z^*} = \lim_{\varepsilon\to 0} \frac{\Phi(\omega+\varepsilon z) - \Phi(\omega)}{\varepsilon} $$

in some sense, however $\Phi(\omega+\varepsilon z)$ is meaningless unless $z \in H$ or Φ is sufficiently regular. For $Z \not\subset H$, thus, the space $\mathbb{D}_{p,1}^{Z}$ consists of functionals which may be "too regular" to be interesting in some applications. However, at the present they seem to be needed for the construction of flows on Wiener space as will be seen in Section 5.

It is straightforward to verify that $\mathcal{S}$ is dense in $L^p(\mu)$ and that the operator ∇ is closeable with respect to $\|\ \|_{p,1;Z}$, with domain $\mathcal{S} \subset L^p(\mu)$ and range in $L^p(\mu; Z)$. The space $\mathbb{D}_{p,1}^{Z}$ can thus be taken to be a dense subset of $L^p(\mu)$, and ∇ has natural bounded linear extension $\nabla^{Z} : \mathbb{D}_{p,1}^{Z} \longrightarrow L^p(\mu; Z^*)$. When no confusion arises, ∇'s superscript may be omitted. In particular, if $\mathcal{P}_n$ is the space of random variables obtained when $C_b^\infty(\mathbb{R}^n)$ is replaced in (2.4) by the family of polynomials in n variables, and $\mathcal{P} = \bigcup_n \mathcal{P}_n$, then $\mathcal{P} \subset \mathbb{D}_{p,1}^{W}$ and (2.5) still holds for any $\Phi \in \mathcal{P}$.

If $(Z_1, \|\ \|_{Z_1}) \hookrightarrow (Z_2, \|\ \|_{Z_2})$ then $\|\Phi\|_{p,1;Z_1} \le \|\Phi\|_{p,1;Z_2}$ and $\mathbb{D}_{p,1}^{Z_2}$ is continuously embedded in $\mathbb{D}_{p,1}^{Z_1}$. In particular $\mathbb{D}_{p,1} = \mathbb{D}_{p,1}^{H}$ is the classical Sobolev space in the Wiener context, and $\mathbb{D}_{p,1}^{W^*}$ is a larger space consisting of Wiener functionals "differentiable only along the W^* directions".

Finally, differentiation can also be defined for random variables taking values in a separable Banach space Y. Let $\mathcal{S}(Y)$ (resp. $\mathcal{P}(Y)$) be $L^\infty(Y)$'s subset of elements having the form $F = \sum_{k=1}^m \Phi_i y_i$, where $m \in \mathbb{N}$, $\Phi_i \in \mathcal{S}$ (resp. $\mathcal{P}$) and $y_i \in Y$, $i = 1, \ldots, m$. The gradient of such an F is defined to be

$$ \nabla F = \sum_{j=1}^m \nabla\Phi_i \otimes y_i \ \in L^\infty(\mu; L(W, Y)). $$

(Here $W^* \otimes Y$ is embedded naturally in $L(W, Y)$ by setting $(l \otimes y)w = \langle w, l\rangle\, y$). We then

define, for a given $W^* \hookrightarrow Z \hookrightarrow W$ as above, $p \in [1,\infty)$ and $F \in \mathcal{S}(Y)$, the Sobolev norms

$$\|F\|_{p,1;Z} \quad = \quad \left(\|F\|^p_{L^p(\mu;Y)} + \|\nabla F\|^p_{L^p(\mu;L(Z,Y))}\right)^{\frac{1}{p}}$$

and $\mathbb{D}^Z_{p,1}(Y)$ will be $\mathcal{S}(Y)$'s completions according to these norms. The same monotonicity relations hold in the differentiation space Z as in the scalar case, and ∇ can be extended to a bounded operator $\nabla^Z : \mathbb{D}^Z_{p,1}(Y) \to L^p(\mu; L(Z,Y))$.

3 The Divergence Operator

3.1 The Divergence of W-valued r.v.'s

The standard definition of the divergence introduces it as an operator on suitable H–valued random variables (cf. e.g. [13], [19]). We now wish to extend it to W–valued random variables. In fact, with no extra effort, the same definition will also apply to W^{**}–valued random variables, one advantage of which is pointed out in Remark 3.15.

Definition 3.1 *For $p \in [1,\infty)$ the space $\mathrm{dom}_p\delta \subset L^p(\mu; W^{**})$ is defined to be the set of all $v \in L^p(\mu; W^{**})$ for which there exists a random variable $\delta v \in L^p(\mu)$, the* **divergence** *of v, such that, for all $\Phi \in \mathcal{S}$,*

$$E\,{}_{W^*}\langle \nabla\Phi, v\rangle_{W^{**}} = E\Phi\delta v. \tag{3.1}$$

In particular,

$$E\delta v = 0 \qquad\qquad \forall v \in \mathrm{dom}_p\delta. \tag{3.2}$$

Moreover, it follows form the L^p duality theory (and $\mathcal{S}$'s density in $L^p(\mu)$) that a necessary and sufficient condition for $v \in \mathrm{dom}_p\delta$ is the existence of a finite positive constant $\gamma = \gamma(v)$ such that for all $\Phi \in \mathcal{S}$

$$E\left|{}_W\langle v, \nabla\Phi\rangle_{W^*}\right| \le \gamma\|\Phi\|_{L^q(\mu)} \tag{3.3}$$

where q is p's conjugate exponent, $\frac{1}{p}+\frac{1}{q}=1$, in which case $\|\delta v\|_{L^p(\mu)}$ is the best possible constant γ in (3.3). In fact, if $p>1$ and $v \in \mathrm{dom}_p\delta$, (3.1) and (3.3) will actually hold for all $\Phi \in \mathbb{D}^W_{q,1}$, in particular for all $\Phi \in \mathcal{P}$.

Remarks 3.2 a) The operator δ and its domain $\mathrm{dom}_p\delta$ are classical objects in the context of H–valued random variables. Obviously, if in our setup v happens to take its values in H a.s. (in

which case the pairing in (3.1) and in (3.3) become $(v, \nabla\Phi)_H$) the definition of δ reduces to the classical one.

b) There are no new deterministic elements in $\mathrm{dom}_1\delta$. Indeed, assume that a (deterministic) $v \in W^{**}$ belongs to $\mathrm{dom}_1\delta$, that is, satisfies (3.3) for all $\Phi \in \mathcal{S}$ and $p=1$. For each $l \in W^*$ denote $\Phi_l = \varphi(\delta l) \in \mathcal{S}$, for some fixed $\varphi \in C_b^\infty(\mathbb{R})$, strictly increasing, (e.g. $\varphi(x) = \arctan(x)$) so that $a := E\varphi'(Z) > 0$, where $Z \sim N(0,1)$. For these test functions, ${}_{W^*}\langle \nabla\Phi_l, v\rangle_{W^{**}} = \varphi'(\delta l)\,{}_{W^*}\langle l, v\rangle_{W^{**}}$, and it thus follows from (3.3) that

$$\sup_{l \in W^*,\ |l|_H = 1} \left| {}_{W^*}\langle l, v\rangle_{W^{**}} \right| \leq \frac{\gamma \, \|\varphi\|_\infty}{a} < \infty. \tag{3.4}$$

This implies that v can be extended as a bouned linear functional on H, i.e. $v \in H$. In other words, there are no deterministic elements in W, or even of W^{**}, that possess a divergence without being in H, and thus already having a divergence in the classical sense.

In view of Remark 3.2b) one might wonder whether there are any non H–valued elements in $\mathrm{dom}_1\delta$ at all. The following example answers this question affirmatively.

Example: For a given ONB $\{e_i\}$ in H, let

$$v = (\delta e_2)e_1 - (\delta e_1)e_2 + (\delta e_4)e_3 - (\delta e_3)e_4 \pm \ldots \quad .$$

By the Ito–Nisio theorem, $v(\omega)$ is a measure preserving transformation of ω and thus isn't supported on H. We claim that v possesses a divergence, which moreover is a.s. 0. This follows from the obvious fact that $\delta\,[\,(\delta e_{2k})e_{2k-1} - (\delta e_{2k-1})e_{2k}\,] = 0$ for all $k \in \mathbb{N}$, and from Lemma 3.4 below which extends this equality to the infinite sum.

It is interesting to note that, on the other hand, $\tilde{v}(\omega) = \omega = \sum_{i=1}^\infty \delta(e_i)\, e_i$, which is (trivially) a measure preserving transformation of ω as well, does *not* have a divergence.

Lemma 3.3 *Let* $\alpha \in \mathbb{D}^W_{p_1,1}$ $\;v \in \mathrm{dom}_{p_2}\delta$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $\alpha v \in \mathrm{dom}_p\delta$ *and*

$$\delta(\alpha v) = \alpha\delta v - {}_{W^*}\langle \nabla\alpha, v\rangle_{W^{**}} \ . \tag{3.5}$$

(As observed in Remark 2.1, the family of α's allowed in this lemma is only slightly more general than $\mathcal{S}$. This result will be applied in Lemma 3.17 at the end of this section).

Proof: For every $\Phi \in \mathcal{S}$

$$\begin{aligned} E\,{}_{W^*}\langle \nabla\Phi, \alpha v\rangle_{W^{**}} &= E\,{}_{W^*}\langle \alpha\nabla\Phi, v\rangle_{W^{**}} = E\,{}_{W^*}\langle \nabla(\alpha\Phi), v\rangle_{W^{**}} - E\,{}_{W^*}\langle \Phi\nabla\alpha, v\rangle_{W^{**}} \\ &= E(\alpha\Phi\delta v) - E\Phi\,{}_{W^*}\langle \nabla\alpha, v\rangle_{W^{**}} = E\Phi\left(\alpha\delta v - {}_{W^*}\langle \nabla\alpha, v\rangle_{W^{**}}\right) \end{aligned}$$

which proves the result. $\square$

Lemma 3.4 *Let* $p\in[1,\infty)$ *and* $\{v_n\}_{n=1}^{\infty}\subset \mathrm{dom}_p\delta$. *If*

i) $v_n \underset{n\to\infty}{\longrightarrow} v$ *weakly in* $L^p(\mu;W)$ *and*

ii) $\{\delta v_n\}_{n=1}^{\infty}$ *is bounded in* $L^p(\mu)$, *i.e.* $\exists M<\infty$ *such that* $\|\delta v_n\|_{L^p(\mu)}\le M$ *for all* n,

then $v\in\mathrm{dom}_p\delta$ *and* $\delta v_n \underset{n\to\infty}{\longrightarrow} \delta v$ *weakly in* $L^p(\mu)$. *In particular* $\|\delta v\|_{L^p(\mu)}\le M$.

Proof: Clearly $v\in L^p(\mu;W)$. Defining q as usual by $\frac{1}{p}+\frac{1}{q}=1$,

$$\left| E_{W}\langle v_n,\nabla\Phi\rangle_{W^*}\right| \le \|\delta v_n\|_{L^p(\mu)}\|\Phi\|_{L^q(\mu)} \le M\|\Phi\|_{L^q(\mu)} \qquad \forall n\in\mathbb{N}, \quad \forall\Phi\in\mathcal{S},$$

so that by (i) v satisfies (3.3) and thus $v\in\mathrm{dom}_p\delta$. Moreover, for every $\Phi\in\mathcal{S}$,

$$E\Phi\delta v = E_{W}\langle v,\nabla\Phi\rangle_{W^*} = \lim_{n\to\infty} E_{W}\langle v_n,\nabla\Phi\rangle_{W^*} = \lim_{n\to\infty} E\Phi\delta v_n. \tag{3.6}$$

Since $\mathcal{S}$ is dense in $L^q(\mu)$ and $\{\delta v_n\}_{n=1}^{\infty}$ is bounded in $L^p(\mu)$, the end terms of (3.6) are equal for all $\Phi\in L^q(\mu)$, in other words $\delta v_n \underset{n\to\infty}{\longrightarrow} \delta v$ weakly in $L^p(\mu)$. $\square$

Corollary 3.5 $\mathbb{D}_{p,1}^{W}(W^{**})\subset\mathrm{dom}_p\delta$ *and* $\delta:\mathbb{D}_{p,1}^{W}(W^{**})\longrightarrow L^p(\mu)$ *is a bounded linear operator.*

Proof: Recall that $\delta:\mathbb{D}_{p,1}^{H}(H)\to L^p(\mu)$ is a bounded linear operator, i.e. there exists a finite constant C such that $\|\delta\Phi\|_{L^p(\mu)}\le C|\Phi|_{p,1}$ for every $\Phi\in\mathbb{D}_{p,1}^{H}(H)$.

Let $F\in\mathbb{D}_{p,1}^{W}(W^{**})$. By definition there exists a sequence $(F_n)_n\subset S(W^{**})$ such that $F_n\to F$ in $\mathbb{D}_{p,1}^{W}(W^{**})$. With no loss of generality we may in fact assume for each n that $\|F_n\|_{p,1}\le\|F\|_{p,1}$ and (since $S(H)$ is dense in $S(W^{**})$ in the $\|\cdot\|_{p,1}$ norm) that $F_n\in S(H)$. Thus

$$\|\delta F_n\|_{L^p(\mu)}\le C|F_n|_{p,1}\le C\|F_n\|_{p,1}\le C\,\|F\|_{p,1} \qquad \forall n\ ,$$

so that by Lemma 3.4 (obviously $F_n\to F$ weakly in $L^p(\mu)$) we conclude that $F\in\mathrm{dom}_p\delta$ and that $\|\delta F\|_{L^p(\mu)}\le C\|F\|_{p,1}$. $\square$

In some cases we have the following convenient approximation. Given a smooth ONB $\mathcal{E}=\{e_i\}_{i=1}^{\infty}$ of H assume $v\in L^p(\mu;W)$ has the representation

$$v=\sum_{i=1}^{\infty}\alpha_i e_i \tag{3.7}$$

that is, $v=\lim_{n\to\infty}v(n)$ in $L^p(\mu;W)$, where $v(n)=\sum_{i=1}^{n}\alpha_i e_i$. Note that $\alpha_i={}_{W}\langle v,e_i\rangle_{W^*}\in L^p(\mu)$ for each i, so that it is possible, for each $n\in\mathbb{N}$, to define the H–valued projections

$$v_n=v_n^{\mathcal{E}}=\sum_{i=1}^{n}E\left(\alpha_i\middle|\mathcal{F}_n\right)e_i=E\left(v(n)\middle|\mathcal{F}_n\right) \tag{3.8}$$

where $\mathcal{F}_n=\mathcal{F}_n^{\mathcal{E}}=\sigma(\delta e_1,\ldots,\delta e_n)$, the sigma–algebra generated by $\delta e_1,\ldots,\delta e_n$, and the conditional expectation on the right is for W–valued random variables.

Proposition 3.6 *Let $p\in[1,\infty)$ and assume that $v\in\mathrm{dom}_p\delta$ is represented as in (3.7). Then*

i) $\lim_{n\to\infty}v_n=v$ *in* $L^p(\mu;W)$

ii) $v_n\in\mathrm{dom}_p\delta$ *and* $\delta v_n=E\left(\delta v\middle|\mathcal{F}_n\right)$

iii) $\lim_{n\to\infty}\delta v_n=\delta v$ *a.s. and in* $L^p(\mu)$.

Proof: The first claim follows from

$$\begin{aligned}\|v-v_n\|_{L^p(\mu;W)} &\le \left\|v-E\left(v\middle|\mathcal{F}_n\right)\right\|_{L^p(\mu;W)}+\left\|E\left(v-v(n)\middle|\mathcal{F}_n\right)\right\|_{L^p(\mu;W)}\\ &\le \left\|v-E\left(v\middle|\mathcal{F}_n\right)\right\|_{L^p(\mu;W)}+\|v-v(n)\|_{L^p(\mu;W)}\,.\end{aligned} \tag{3.9}$$

The first term converges to zero by the (W–valued) martingale L^p convergence theorem, while the second term does so by assumption.

Next, note that $E\left(\Phi\middle|\mathcal{F}_n\right)\in\mathcal{S}$ for an arbitrary $\Phi\in\mathcal{S}$ and $\nabla E\left(\Phi\middle|\mathcal{F}_n\right)=E\left(\nabla\Phi\middle|\mathcal{F}_n\right)$. Then

$$\begin{aligned}E\,{}_{W}\langle v_n,\nabla\Phi\rangle_{W^*} &= E\,{}_{W}\langle v(n),E\left(\nabla\Phi\middle|\mathcal{F}_n\right)\rangle_{W^*}=E\,{}_{W}\langle v(n),\nabla E\left(\Phi\middle|\mathcal{F}_n\right)\rangle_{W^*}\\ &= E\,{}_{W}\langle v,\nabla E\left(\Phi\middle|\mathcal{F}_n\right)\rangle_{W^*}=E\left(\delta v\,E\left(\Phi\middle|\mathcal{F}_n\right)\right)=E\left(E\left(\delta v\middle|\mathcal{F}_n\right)\Phi\right)\end{aligned}$$

which proves (ii).

Finally (iii) follows from (ii) by the martingale convergence theorem applied to $\delta v_n = E\left(\delta v \middle| \mathcal{F}_n\right)$ in both the a.s. and L^p senses. $\square$

In Section 5 we shall need the following extension of the Proposition's first statement to v's parametrized by some positive measure space $(I, \mathcal{I}, \lambda)$.

Corollary 3.7 *For a given* $p \in [1, \infty)$ *assume that* $v_t(\omega) \in L^p(I \times W, \lambda \times \mu\,; W)$ *satisfies*

$$\lim_{n\to\infty} \int_I E \left\| v_t - \sum_{i=1}^n \alpha_{i_t} e_i \right\|_W^p d\lambda(t) = 0$$

for some smooth ONB $\mathcal{E} = \{e_i\}_{i=1}^\infty$ *of* H *(with* $\alpha_{i_t} = {}_W\langle e_i, v_t \rangle_{W^*}$*). Then*

$$\lim_{n\to\infty} \int_I E \left\| v_t - \sum_{i=1}^n E\left(\alpha_{i_t} \middle| \mathcal{F}_n\right) e_i \right\|_W^p d\lambda(t) = 0.$$

Its proof essentially repeats the one of Proposition 3.6*i*) except that one must add a dominated convergence argument (to the integral over I) for the first term in (3.9)'s appropriate extension to converge to zero.

Definition 3.8 *Let* $p \in [1, \infty)$.

a) An element $v \in \mathrm{dom}_p \delta$ *is said to be* divergence-free *if* $\delta v = 0$, *i.e. if* $E\,{}_{W^*}\langle \nabla \Phi, v \rangle_{W^{**}} = 0$ *for all* $\Phi \in \mathcal{S}$. *The class of all such divergence-free is denoted by* $\mathrm{dom}_p^0 \delta$.

b) An element $u \in L^p(\mu; H) \subset L^p(\mu; W)$ *is said to be* exact *if there exists a* $\Psi \in \mathbb{D}_{p,1}^H$ *such that* $u = \nabla \Psi$. *This class of exact* H*–valued random variables is denoted* $L_{\mathrm{e}}^p(\mu; H)$.

Lemma 3.9 *Let* $p \in [1, \infty)$.

a) If $u \in \mathrm{dom}_p^0 \delta$ *then* $(1+\mathcal{L})^{-\beta} u \in \mathrm{dom}_p^0 \delta$ *for every* $\beta \geq 0$.

b) $\mathrm{dom}_p^0 \delta \cap L_{\mathrm{e}}^p(\mu; H) = \{0\}$

Proof: **a)** By (2.2) $(1+\mathcal{L})^{-\beta} u \in L^p(\mu; W^{**})$. By (2.2), (2.3) and T_t's self-adjointness, for any $\Phi = \varphi(\delta e_1, \ldots, \delta e_n) \in \mathcal{S}$, $\nabla \Phi = \sum_1^n \varphi_i e_i \in \mathcal{S}(W^*)$, (here $\varphi_i = \frac{\partial \varphi}{\partial x_i}(\delta e_1, \ldots, \delta e_n)$),

and

$$
\begin{aligned}
E_{W^*}\langle \nabla\Phi, (1+\mathcal{L})^{-\beta}u\rangle_{W^{**}} &= \frac{1}{\Gamma(\beta)}\, E\int_0^\infty t^{\beta-1}e^{-t}\;{}_{W^{**}}\langle T_t u, \sum_1^n \varphi_i e_i\rangle_{W^*}\, dt \\
&= \frac{1}{\Gamma(\beta)}\, E\int_0^\infty t^{\beta-1}e^{-t}\sum_1^n \Big({}_{W^{**}}\langle T_t u, e_i\rangle_{W^*}\, \varphi_i\Big)\, dt \\
&= \frac{1}{\Gamma(\beta)}\, E\int_0^\infty t^{\beta-1}e^{-t}\sum_1^n \Big(T_t\langle u, e_i\rangle\Big)\, \varphi_i\, dt \\
&= \frac{1}{\Gamma(\beta)}\, \int_0^\infty t^{\beta-1}e^{-t}\sum_1^n E\Big(T_t\varphi_i\langle u, e_i\rangle\Big)\, dt \\
&= E_{W^*}\langle (1+\mathcal{L})^{-\beta}\nabla\Phi, u\rangle_{W^{**}} \\
&= E_{W^*}\langle \nabla\mathcal{L}^{-\beta}\Phi, u\rangle_{W^{**}} \qquad \text{(by (iv) in subsection 2.2)} \\
&= 0\,.
\end{aligned}
$$

b) If $u=\nabla\Psi$ and $\delta u=0$, then $0=\delta\nabla\Psi=\mathcal{L}\Psi$. Thus $\Psi=E\Psi$ a.s., so that $u=0$. $\square$

The following proposition essentially states that the only "new" W–valued vector fields with divergence are divergence free.

Proposition 3.10 *Let $p\in[1,\infty)$. Each $v\in \mathrm{dom}_p\delta$ can be uniquely decomposed as a sum $v=v^0+v_{\mathrm{e}}$ where v^0 is divergence free and v_{e} is exact (with divergence). Equivalently*

$$
\mathrm{dom}_p\delta = \mathrm{dom}^0\delta \oplus \left(L^p_{\mathrm{e}}(\mu;H)\cap \mathrm{dom}_p\delta\right).
$$

Proof: Let $v\in \mathrm{dom}_p\delta$. By (3.2) and the remarks following (2.2), $v_e=\nabla\left(\mathcal{L}^{-1}\delta v\right)$ is a well defined element of $L^p_{\mathrm{e}}(\mu;H)\cap\mathrm{dom}_p\delta$. In order to prove the decomposition, we need to check that $v^0:=(v-v_e)\in\mathrm{dom}^0_p\delta$. Indeed, as remarked above, $v_e\in\mathrm{dom}_p\delta$. Moreover, by (ii) of subsection (2.1),

$$
\delta v^0 = \delta v - \delta\nabla\left(\mathcal{L}^{-1}\delta v\right) = 0.
$$

The uniqueness follows directly from Lemma 3.9b. $\square$

Remark 3.11 Heuristically, vector fields which possess divergence generate flows. This will be formalized, under appropriate assumptions, in Section 5 noting in addition that the flows generated

by divergence free vector fields are measure preserving ("rotations") while those generated by $H-$valued vector fields ("shifts") have been already studied, for example in [3], [16] and [21]. What Proposition 3.10 suggests is that a general vector field which generates flows can be decomposed into a "rotation" generating component and a "shift" generating component.

The formula for the classical divergence's second moment has its counterpart for $W-$valued variables as well, but since it involves operators and their divergence, it is deferred till the next subsection (Lemma 3.17).

3.2 The Divergence of Operator Valued r.v.'s

In the next section divergence–free vector fields will be characterized as the divergence of an antisymmetric operator. This is, at this stage, only a formal declaration. The remainder of this section is dedicated, therefore, to precise what is meant by the divergence of an operator and to present some of its properties.

Indeed, the classical divergence is defined for random variables taking values not only in H but in H's tensor powers as well. Thus, for example, the divergence $\boldsymbol{\delta}\mathbf{A}$ of a random Hilbert Schmidt operator $\mathbf{A}(\omega)$ in H is characterized by

$$E(\mathbf{A}, \nabla F)_{H^{\otimes 2}} = E(\boldsymbol{\delta}\mathbf{A}, F)_H \qquad \forall F \in \mathcal{S}(H). \tag{3.10}$$

Here $(\mathbf{A}, \mathbf{B})_{H^{\otimes 2}} = \mathrm{tr}\mathbf{A}\mathbf{B}^{\mathrm{T}} = \sum_{i=1}^{\infty}(\mathbf{A}e_i, \mathbf{B}e_i)_H$ (for any ONB $\{e_i\}$) is the natural inner product of two Hilbert Schmidt operators on H.

The random operators we wish to generalize the divergence $\boldsymbol{\delta}$ to will have W^* as their domain and a fixed arbitrary Banach space as their range (instead of $\mathbb{R}$ as in subsection 3.1). To carry out this generalization, recall first the definition of the trace $\mathrm{tr}\, T$ of an operator $\mathbf{T} \in L(W^*, W^{**})$, namely $\sum_{i=1}^{\infty} {}_{W^*}\langle e_i, \mathbf{T}e_i\rangle_{W^{**}}$ if this sum exists for every smooth ONB (e_i) of H, and is the same for all such bases. In particular, every finite rank operator from W^* *to* W^{**} has a trace.

Henceforth, Y will be a fixed Banach space, $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 3.12 *Define* $\mathbf{dom}_{p,Y}\boldsymbol{\delta} = \mathbf{dom}_p\boldsymbol{\delta}$ *to be the set of all* $\mathbf{K} \in L^p(\mu; L(W^*, Y))$ *for which there exists a* $\boldsymbol{\delta}\mathbf{K} \in L^p(\mu; Y)$, *the* **divergence** *of* $\mathbf{K}$, *such that*

$$E\,\mathrm{tr}\left(\mathbf{K}^T \nabla^{W^*} F\right) = E_Y\langle \delta\mathbf{K}, F\rangle_{Y^*} \tag{3.11}$$

for all $F \in \mathcal{S}(Y^*)$.

Remark 3.13 In (3.11), $\nabla^{W^*} F$ has finite rank a.s., so that its left hand side makes sense. If $\mathbf{K}$ itself has a deterministic finite dimensional range, (3.11) will also hold for all $F \in \mathbb{D}^{W^*}_{q,1}(Y^*)$. Indeed, both of its terms pass to the limit when $F_n \to F \quad (F_n \in \mathcal{S}(Y^*))$.

The uniqueness of $\boldsymbol{\delta}\mathbf{K}$ is a consequence of $S(Y^*)$'s density in $L^q(\mu; Y^*)$. Moreover, if $\mathbf{K} \in \mathbf{dom}_{p,Y}\boldsymbol{\delta}$ and Y is continously embedded in another Banach space Y_1, then (3.11) obviously holds for all $F \in \mathcal{S}(Y_1^*) \subset \mathcal{S}(Y^*)$. Thus $\mathbf{dom}_{p,Y}\boldsymbol{\delta} \subset \mathbf{dom}_{p,Y_1}\boldsymbol{\delta}$ and $\boldsymbol{\delta}\mathbf{K}$ is the same Y–valued random variable whether $\mathbf{K}$'s range is taken to be Y or Y_1. That is the reason why it isn't necessary to include the subscript Y in the notation of $\boldsymbol{\delta}$.

Just as in the scalar case a necessary and sufficient condition for $\mathbf{K} \in \mathbf{dom}_{p,Y}\boldsymbol{\delta}$ is the existence of a finite positive constant $\gamma = \gamma(\mathbf{K})$ such that for all $F \in \mathcal{S}(Y^*)$

$$|E \,\mathrm{tr}\left(\mathbf{K}^T \nabla^{W^*} F\right)| \leq \gamma \|F\|_{L^q(\mu; Y^*)} \tag{3.12}$$

in which case $\|\boldsymbol{\delta}\mathbf{K}\|_{L^p(\mu;Y)}$ is the best possible constant γ in (3.12).

We denote $\mathbf{dom}_p\boldsymbol{\delta} = \mathbf{dom}_{p,W^{**}}\boldsymbol{\delta}$. Indeed, in Sections 4 and 5, Y will typically be W^{**} and together with $\delta\mathbf{K}$ we shall need to consider $\delta\mathbf{K}^T$ as well. (By a slight abuse of notation, $\mathbf{K}^T$ actually stands for $\mathbf{K}^T\big|_{W^*}$). Recall that $L(W^*, W^{**})$ can be also seen as the space $M_2(W^*)$ of bilinear forms in W^*, and in this interpretation, $\mathbf{K}^T(l_1, l_2) = \mathbf{K}(l_2, l_1)$). In particular, then, $\boldsymbol{\delta}$'s domain contains $W \otimes W$ in this case, as stated in the abstract.

A useful connection between this divergence $\boldsymbol{\delta}$ and its scalar counterpart δ is the following

Lemma 3.14 *An element* $\mathbf{K} \in L^p(\mu; L(W^*, Y^{**})$ *belongs to* $\mathbf{dom}_{p,Y^{**}}\boldsymbol{\delta}$ *if and only if* $\mathbf{K}^T l \in \mathrm{dom}_p\delta$ *for every* $l \in Y^* \subset Y^{**}$ *and for some* $C > 0$

$$\|\boldsymbol{\delta}\left(\mathbf{K}^T l\right)\|_{L^p(\mu)} \leq C \|l\|_{Y^*} \qquad \forall\, l \in Y^*. \tag{3.13}$$

In this case

$$\delta(\mathbf{K}^T l) = {}_{Y^*}\langle l, \boldsymbol{\delta}\mathbf{K}\rangle_{Y^{**}} \qquad \text{a.s.} \tag{3.14}$$

and more generally, for any $F \in \mathcal{S}(Y^*)$, $\quad \mathbf{K}^T F \in \mathrm{dom}_p\delta \quad$ *and*

$$\delta(\mathbf{K}^T F) = {}_{Y^*}\langle F, \boldsymbol{\delta}\mathbf{K}\rangle_{Y^{**}} - \mathrm{tr}(\mathbf{K}^T \nabla^{Y^*} F) \tag{3.15}$$

Proof: Throughout the proof, ∇ will stand for ∇^{Y^*}. First assume that $\boldsymbol{\delta}\mathbf{K}$ exists. For any $\Phi\in\mathcal{S}$, $l\in Y^*$ and denoting $G=\Phi l\in\mathcal{S}(Y^*)$, it is straightforward to verify that

$$\operatorname{tr}(\mathbf{K}^T\nabla G) = {}_{W^*}\langle\nabla\Phi,\mathbf{K}^T l\rangle_{W^{**}}. \tag{3.16}$$

Then

$$E\,{}_{W^*}\langle\nabla\Phi,\mathbf{K}^T l\rangle_{W^{**}} = E\operatorname{tr}(\mathbf{K}^T\nabla G) = E\,{}_{Y^*}\langle G,\boldsymbol{\delta}\mathbf{K}\rangle_{Y^{**}} = E\Phi\,{}_{Y^*}\langle l,\boldsymbol{\delta}\mathbf{K}\rangle_{Y^{**}}$$

and $\Phi\in\mathcal{S}$ being arbitrary, $\boldsymbol{\delta}(\mathbf{K}^T l)$ exists and (3.14) holds, from which (3.13) follows directly. In the converse direction, it follows from (3.13) that there exists a $\Delta_{\mathbf{K}}\in L(Y^*,L^p(\mu))\approx L^p(\mu;Y^{**})$ (we shall indeed relate to $\Delta_{\mathbf{K}}$ as an element of $L^p(\mu;Y^{**})$) such that for all $l\in Y^*$

$$\boldsymbol{\delta}(\mathbf{K}^T l) = {}_{Y^*}\langle l,\Delta_{\mathbf{K}}\rangle_{Y^{**}} \qquad \text{a.s.}, \tag{3.17}$$

so that, for any $F=\sum_{j=1}^m\Phi_j l_j\in\mathcal{S}(Y^*)$

$$\begin{aligned}
E\operatorname{tr}(\mathbf{K}^T\nabla F) &= \sum_{j=1}^m \operatorname{tr}\mathbf{K}^T\nabla(\Phi_j\,l_j) = \sum_{j=1}^m E\,{}_{W^*}\langle\nabla\Phi_j,\mathbf{K}^T l_j\rangle_{W^{**}}\\
&= \sum_{j=1}^m E\delta(\mathbf{K}^T l_j)\Phi_j = E\sum_{j=1}^m {}_{Y^*}\langle l_j,\Delta_{\mathbf{K}}\rangle_{Y^{**}}\Phi_j\\
&= \Phi\,{}_{Y^*}\langle F,\Delta_{\mathbf{K}}\rangle_{Y^{**}}.
\end{aligned}$$

Thus $\boldsymbol{\delta}\mathbf{K}$ exists by definition and is actually $\Delta_{\mathbf{K}}$, (3.17) being nothing else but (3.14). Turning to (3.15), and with $F=\Phi\,l$, ($\Phi\in\mathcal{S}$ and $l\in Y^*$), it follows from Lemma 3.3 that

$$\delta(\mathbf{K}^T F) \;=\; \delta(\Phi\mathbf{K}^T l) \;=\; \Phi\,\delta(\mathbf{K}^T l) - {}_{W^*}\langle\nabla\Phi,\mathbf{K}^T l\rangle_{W^{**}}$$

which proves the claim in view of (3.14) and (3.16). $\square$

Remark 3.15 This lemma might shed some light on two questions concerning the rôle of Y^{**} in general and W^{**} in particular. First, the left hand side of (3.14) requires the (scalar) divergence δ to be defined for random variables taking values in W^{**}, not only in W. Secondly, the vector divergence $\boldsymbol{\delta}\mathbf{K}$ of $\mathbf{K}$ (initially identified in the proof as $\Delta_{\mathbf{K}}$) must be allowed to take values in Y^{**}, not only in Y, for the lemma to hold.

Corollary 3.16 *For any* $p\geq 1$

a) $\mathbb{D}^W_{p,1}(L(W^*,W^{**}))\subset \mathbf{dom}_p\boldsymbol{\delta}$.

b) If $v\in \mathrm{dom}_p\delta$ *and* $y\in W^{**}$ *then* $v\otimes y\in \mathbf{dom}_p\boldsymbol{\delta}$ *and* $\boldsymbol{\delta}(v\otimes y)=\delta(v)y$.

c) Let $\alpha\in\mathbb{D}^W_{p_1,1}$, $\mathbf{A}\in\mathbf{dom}_{p_2}\boldsymbol{\delta}$ *and* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. *Then* $\alpha\mathbf{A}\in \mathrm{dom}_p\delta$ *and*

$$\boldsymbol{\delta}\alpha\mathbf{A}=\alpha\boldsymbol{\delta}\mathbf{A}-\mathbf{A}\nabla^{W^*}\alpha \qquad \text{a.s.} \tag{3.18}$$

Proof: Let $l\in W^*$. If $\mathbf{K}\in\mathbb{D}^W_{p,1}(L(W^*,W^{**}))$ it is straightforward to verify that the same is true for $\mathbf{K}^T$, and therefore that $\mathbf{K}^T l\in\mathbb{D}^W_{p,1}(W^{**})$, so that $\mathbf{K}l\in\mathrm{dom}_p\delta$ by Corollary 3.5. Since $l\in W^*$ was arbitrary, a) follows from Lemma 3.14. As for b), for any $l\in W^*$

$$\begin{aligned} {}_{W^*}\langle l,\boldsymbol{\delta}v\otimes y\rangle_{W^{**}} &= \delta\left((v\otimes y)^T l\right) = \delta({}_{W^*}\langle l,y\rangle_{W^{**}}v)\\ &= {}_{W^*}\langle l,y\rangle_{W^{**}}\delta v = {}_{W^*}\langle l,(\delta v y)\rangle_{W^{**}} \end{aligned}$$

which proves the claim.

Finally for c), assume without loss of generality that $\alpha\in\mathcal{S}$. and let $l\in W^*$. Denoting $F=\alpha\, l$, it follows from (3.15) that

$${}_{W^*}\langle l,\delta(\alpha\mathbf{A})\rangle_{W^{**}}=\delta(\mathbf{A}^T F)={}_{W^*}\langle \alpha l,\delta\mathbf{A}\rangle_{W^{**}}-{}_{W^*}\langle l,\mathbf{A}\nabla\alpha\rangle_{W^{**}}$$

from which (3.18) follows, again since $l\in W^*$ was arbitrary. $\square$

We conclude this section with an extension of a classical second moment identity.

Lemma 3.17 *Let* $G\in\mathcal{S}(W^*)$ *and* $u\in\mathbb{D}^{W^*}_{1,1}(W^{**})$. *Then*

$$E(\delta u\delta G)=E_{W^*}\langle G,u\rangle_{W^{**}}+E\mathrm{tr}\left(\nabla G\,\nabla^{W^*}u\right). \tag{3.19}$$

If, moreover, $\nabla u\in\mathbf{dom}_1\boldsymbol{\delta}$ *it will also hold that*

$$E(\delta u\delta G)=E_{W^*}\langle G,u\rangle_{W^{**}}+E_{W^*}\langle G,\delta(\nabla u)^T\rangle_{W^{**}} \tag{3.20}$$

Proof: Clearly $\alpha:=\delta G$ belongs to $\mathcal{P}$, so that by Lemma 3.3

$$\begin{aligned} E(\delta u\delta G) &= E\delta(\delta G\, u)+E_{W^*}\langle\nabla\delta G,u\rangle_{W^{**}}\\ &= 0 + E_{W^*}\langle G,u\rangle_{W^{**}}+E_{W^*}\langle\delta((\nabla G)^T),u\rangle_{W^{**}}. \end{aligned}$$

In the last equality we have used the well known identity $\nabla\delta G = G + \delta((\nabla G)^T)$ (in which the third term is, in fact, W^*–valued). The identity (3.19) will then follow by applying (3.11) to the last term, together with Remark 3.13 (here, $Y = W^*$). A second application of (3.11) yields (3.20), this time with $Y = W^{**}$ and thinking of G as an element of $\mathcal{S}(W^{***})$. $\square$

4 Divergence–free W–valued random variables

In this section we study the structure of $\mathrm{dom}_p^0\delta$. In classical analysis, zero–mean divergence–free vector fields generate rotations. On the other hand, the tangent space of the special orthogonal group $SO(n)$ can be identified with the space of skew symmetric $n\times n$ matrices. Building on this correspondence, a W–valued random variable u was associated in [22] with each sufficiently smooth random skew symmetric bounded operator $\mathbf{A}(\omega)$ in H, by

$$v = \sum_{i=1}^{\infty} \delta(\mathbf{A}e_i)e_i \tag{4.1}$$

assuming the series converges in $L^p(\mu; W)$. Here $\mathcal{E} = \{e_i\}_{i=1}^{\infty}$ is a given smooth ONB. Under further smoothness and moment assumptions these W–valued random variables (vector fields) were then shown in [22] to induce invariant flows in W, suggesting that they are, in our language, divergence–free. Note that (4.1) can now be written as $v = \boldsymbol{\delta}\mathbf{A}^T$. Indeed, for any $l \in W^*$,

$${}_{W^*}\langle l, v\rangle_{W^{**}} = \sum_{i=1}^{\infty} \delta(\mathbf{A}e_i)(l, e_i)_H = \delta\left(\sum_{i=1}^{\infty} \mathbf{A}e_i(l, e_i)_H\right) = \delta\left(\mathbf{A}\sum_{i=1}^{\infty}(l, e_i)_H\, e_i\right) = \delta(\mathbf{A}l)$$

which by Lemma 3.14 implies that $v = \delta\mathbf{A}^T$, since $l \in W^*$ was arbitrary.

We show here (Theorem 4.2) that *every* divergence-free W–valued v can be obtained in this way, for some suitable skew symmetric random bilinear form $\mathbf{A}$ on W^*.

Lemma 4.1 *Let $u \in \mathrm{dom}_1^0\delta$ be such that $(\nabla u)^T \in \mathbf{dom}_1\boldsymbol{\delta}$. Then $u = -\delta\,(\nabla u)^T$.*

Proof: For any given $G \in \mathcal{S}(W^*)$ apply Lemma 3.17 to obtain

$$E\,{}_{W^*}\langle G, u\rangle_{W^{**}} + E\,{}_{W^*}\langle G, \delta(\nabla u)^T\rangle_{W^{**}} = 0$$

from which the result follows, since G was arbitrary. $\square$

Theorem 4.2 *Let $v\in L^p(\mu;W^{**})$, $p\in[1,\infty)$. Then $v\in \mathrm{dom}_p^0\delta$ iff there exists an $\mathbf{A}\in\mathbf{dom}_p\boldsymbol{\delta}$ with $\mathbf{A}+\mathbf{A}^T=0$ such that $v=\boldsymbol{\delta}\mathbf{A}$.*

Proof: Assume first that there exists an $\mathbf{A}$ as stated, and let $\Phi=\varphi(\delta e_1,\ldots,\delta e_n)\in\mathcal{S}_n$. Then, using the notation $\partial_x\Phi={}_W\langle x,\nabla\Phi\rangle_{W^*}$ for $\Phi\in\mathcal{S}$ and $x\in W$, (and similarly $\partial^2_{xy}\Phi$),

$$
\begin{aligned}
E_{W^*}\langle\nabla\Phi,v\rangle_{W^{**}} &= E_{W}\langle\sum_{i=1}^n\partial_{e_i}\varphi(\delta e_1,\ldots,\delta e_n)e_i,\boldsymbol{\delta}\mathbf{A}\rangle_{W^*}\\
&\overset{(3.14)}{=} \sum_{i=1}^n E\partial_{e_i}\varphi(\delta e_1,\ldots,\delta e_n)\delta(\mathbf{A}^Te_i)\\
&= \sum_{i=1}^n E_W\langle\nabla\partial_{e_i}\varphi,\mathbf{A}^Te_i\rangle_{W^*}\\
&= E\sum_{i,j=1}^n\partial^2_{e_ie_j}\Phi_{W^*}\langle e_j,\mathbf{A}^Te_i\rangle_{W^{**}} \;=\; 0 \qquad (4.2)
\end{aligned}
$$

because $\mathrm{tr}AB=0$ for all symmetric (respectively skew symmetric) $n\times n$ matrices A and B. Since $\Phi\in\mathcal{S}$ was arbitrary, it follows by definition that δv exists and is 0.

For the converse, denote $u=(1+\mathcal{L})^{-1}v$ which also belongs to $\mathrm{dom}_p^0\delta$ by Lemma 3.9a). It then follows from Lemma 4.1 that

$$v-\mathcal{L}(1+\mathcal{L})^{-1}v=(1+\mathcal{L})^{-1}v=-\delta(\nabla(1+\mathcal{L})^{-1}v)^T.$$

Recalling that $\mathcal{L}=\delta\nabla$, we thus have $v=\delta\left(\nabla(1+\mathcal{L})^{-1}v-(\nabla(1+\mathcal{L})^{-1}v)^T\right)$. □

Remark 4.3 Proposition 3.10 and Theorem 4.2 combined provide a unique decomposition of any element of $\mathrm{dom}_p\delta$ as a sum of the gradient of a scalar random variable and the divergence of a random antisymmetric bilinear form. This constitutes an extension of I. Shigekawa's first order Hodge–Kodaira theorey in Wiener space (cf. [17]).

5 Wiener space valued vector fields and their induced flows

One of the main interests in W–valued random variables v is the possible existence of their generated quasiinvariant flows. More generally, we shall consider time–dependent vector fields and throughout, $I=[a,b]$ will be an compact interval in $\mathbb{R}$.

Definition 5.1 *A measurable mapping $v : I\times W \longrightarrow W$ will be said to generate a flow $T=\{T_{s,t},\ s,t\in I\}$ if T is jointly measurable and*

i) $$T_{s,t}(\omega)=\omega+\int_s^t v_r(T_{s,r}(\omega))\,dr, \qquad \forall s,t\in I, \quad \text{a.s.} \tag{5.1}$$

ii) $$T_{r,t}(\omega)=T_{s,t}(T_{r,s}(\omega)) \qquad \forall r,s,t\in I \quad \text{a.s.} \tag{5.2}$$

iii) $$T_{s,t}\mu\ll\mu, \quad \textit{where} \quad \Lambda_{s,t}=dT^*_{s,t}\mu/d\mu \qquad \forall s,t\in I. \tag{5.3}$$

T will be said to be the **unique** *flow generated by v if whenever $S=\{S_{s,t},\ s,t\in I\}$ satisfies (i)–(iii) as well, $S_{s,t}=T_{s,t}$ a.s. for all $s,t\in I$.*

In [3] A. Cruzeiro proved the existence of a flow for H–valued (time independent) vector fields whose gradient and divergence have finite exponential moments, and G. Peters noted in [16] that the weaker operator norm of ∇v could be used in the finite exponential moment assumption (instead of the usual Hilbert-Schmidt norm). In [21, Section 5.3] the result of Cruzeiro was extended and its assumptions relaxed.

Based on Cameron–Martin's theorem on constant shifts and its subsequent generalizations, H–valued vector fields are indeed natural candidates to generate quasiinvariant flows. However, it is obvious that quasiinvariant shifts exist which do not necessarily act along H.

In this section we present two general results. The first , Theorem 5.3, states that "representable" (to be defined) W–valued time dependent vector field $\{v_t,\ t\in I\}$, whose gradients and divergences satisfy some standard exponential moment conditions, generates a quasiinvariant flow. As such it extends [9, 22]. The decomposition of Proposition 3.10 represent v's H–shift and rotation components respectively.

The second result, Theorem 5.7, shows that these are essentially all the W–valued random variables which do so, thus providing a qualitative description of the Wiener tangent space.

Definition 5.2 *Let $\mathcal{E}=\{e_i\}_{i=1}^\infty$ be a given smooth ONB of H.*

a) Denote $\pi_n(\omega)=\sum_{i=1}^n {}_W\langle\omega,e_i\rangle_{W^*}e_i$, $\quad W_n=\pi_n(W) \quad$ and $\quad \mu_n=\pi_n^*\mu$ (without writing the dependence on $\mathcal{E}$ explicitly). Moreover, denote $\mathcal{F}_n=\sigma(\pi_n)$ and $E_n(\cdot)=E(\cdot\,|\,\mathcal{F}_n)$. A (not necessarily scalar) random variable on $(W,\mathcal{F},\mu)$ will be said to be **cylindrical** *if it is $\mathcal{F}_n$–measurable for some $n\in\mathbb{N}$.*

*b) A time-dependent vector field $\mathbf{v}=\{v_t\}_{t\in I}\in L^1(\text{leb}\times\mu;W)$ is said to be $\mathcal{E}$–***representable** *if there exist $\mathbf{v}_i=\{v_{i,t}\}_{t\in I}\in L^1(\text{leb}\times\mu)$, $i\in\mathbb{N}$, such that $v_t=\sum_{i=1}^\infty v_{i,t}e_i$ in $L^1(\text{leb}\times\mu;W)$.*

Theorem 5.3 *Given a smooth ONB* $\mathcal{E}=\{e_i\}_{i\in\mathbb{N}}$ *let* $\mathbf{v}=\{v_t\}_{t\in I}$ *be an* $\mathcal{E}$-**representable** *vector field such that* $v_t\in\mathrm{dom}_1\delta$ *for all* $t\in I$*, and which moreover possesses a jointly measurable decomposition* $v_t=u_t+B_t$*,* $t\in I$*, for which*

(i) $u_t\in\mathbb{D}_{1,1}(H)$ *and* $B_t\in\mathbb{D}^W_{1,1}(W)$ *with* $\delta B_t=0$*, for all* $t\in I$

($\mathbb{D}^W_{1,1}(W)$ *is defined at the end of subsection 2.2)*

(ii) $\exists\theta>0$ *such that*

$$\Gamma_H(\theta) \;:=\; E\int_a^b \exp\theta\left\{\|\nabla^H u_t\|_{L(H)}+|\delta u_t|\right\}dt<\infty \qquad \text{and} \tag{5.4}$$

$$\Gamma_W(\theta) \;:=\; \sup_n E\int_a^b \exp\theta\|\pi_n\nabla^W B_t\|_{L(W)}\,dt<\infty. \tag{5.5}$$

Then $\mathbf{v}$ *generates a flow* $\{T_{s,t},\ s,t\in I\}$ *with*

$$\frac{dT^*_{s,t}\mu}{d\mu}=\exp\left\{\int_s^t \delta v_r(T_{t,r})\,dr\right\} \tag{5.6}$$

and for all $p>1$ *and* $|t-s|<\frac{\theta}{2p}$*,*

$$E\left(\frac{dT^*_{s,t}\mu}{d\mu}\right)^p\le e^{1/p}\left(1+\frac{2p-2}{\theta}\sqrt{\Gamma_H(\theta)\Gamma_W(\theta)}\right). \tag{5.7}$$

If, in addition, the paths $t\to v_t$ *are a.s. continuous then the flow* $\{T_{s,t},\ s,t\in I\}$ *is unique.*

Remarks 5.4

(a) The decomposition assumed in Theorem 5.3 is not unique. In particular, it is not necessarily the one provided by Proposition 3.10.

(b) It follows from (5.5) by Fatou's lemma that

$$E\int_a^b \exp\theta\|\nabla^W B_t\|_{L(W)}\,dt\le\Gamma_W(\theta)<\infty, \tag{5.8}$$

however the proof below needs the less elegant assumption (5.5) itself. If $\mathcal{E}$ is a Schauder basis of W, then (5.5) is also *implied* by (5.8) (possibly with a smaller θ), but in general this need not be the case.

(c) Equation (5.4) and (5.8) actually imply that $u_t\in\mathbb{D}_{p,1}(H)$ and $B_t\in\mathbb{D}^W_{p,1}(W)$ for all $t\in I$ and $p\ge1$. However the initial assumption in (i) for $p=1$ was needed to give meaning to $\nabla^H u_t$ and $\nabla^W B_t$ in the first place.

(d) It follows from (5.6) that if v_r is divergence free for all r then the flow is measure preserving.

Before proceeding with the proof, let us recall some relevant results pertaining to jointly measurable time dependent vector field $\eta = \{\eta_t(x)\}_{t\in I}$ on W_n as stated, for example, in [21, Section 5.1]. Assume that η is locally integrable in the t variable, C^1 in the x variable and that for some $\theta > 0$

$$\Gamma^\eta(\theta) := E_n \int_a^b \exp\left\{\theta\left(\|\nabla\eta_t\|_{L(W_n)} + |\delta\eta_t|\right)\right\} dt < \infty. \tag{5.9}$$

Then η generates a quasiinvariant flow $\{X_{s,t},\ s,t\in I\}$ whose Radon-Nikodym derivative

$$R_{s,t} := \frac{d\left(X^*_{s,t}\mu_n\right)}{d\mu_n} = \exp\int_s^t (\delta\eta_r)\,(X_{t,r})\,dr$$

satisfies, for any $p>1$ and for all $s,t\in I$ for which $|t-s| < \frac{\theta}{p}$,

$$E_n\,(R_{s,t})^p \le e^{1/p}\left(1 + \tfrac{p-1}{\theta}\,\Gamma^\eta(\theta)\right) \tag{5.10}$$

(This bound is essentially given in [21, Theorem 5.1.3]). Indeed, with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned} E_n\,(R_{s,t}(x))^p &= E_n\,(R_{s,t}(X_{s,t}x))^{p-1} = E_n e^{(p-1)\int_s^t \delta\eta_r(X_{t,r}X_{s,t}x)\,dr} \\ &\le 1 + \frac{p}{\theta}\int_I E_n e^{\frac{\theta}{q}|\delta\eta_r(x)|} R_{s,r}(x)\,dr \\ &\le \left(1 + \frac{p-1}{\theta}\int_I E_n e^{\theta|\delta\eta_r(x)|} dr\right) + \frac{1}{\theta}\int_s^t E_n\,(R_{s,r}(x))^p\,dr\ . \end{aligned} \tag{5.11}$$

The second row follows from $e^{\int_\alpha^\beta f} \le 1 + \frac{1}{\gamma}\int_{\alpha\wedge\beta}^{\alpha\vee\beta} e^{\gamma f}$ whenever $|\beta-\alpha|<\gamma$ and $f(x)\ge 0$ (here $\gamma=\frac{\theta}{p}$), while the third row follows from Young's inequality $yz\le\frac{y^p}{p}+\frac{z^q}{q}$, $y,z>0$. The estimate (5.10) now follows by by applying Gronwall's inequality to (5.11). Note that for the purposes of (5.10), $\Gamma^\eta(\theta)$ could have been defined in (5.9) without the gradient term.

Proof of Theorem 5.3 Consider v's finite dimensional "projections" $v_t^{(n)} = E_n(\pi_n(v_t))$, $t\in I$, which may be written as $v_t^{(n)} = \widetilde{v}_t^{(n)}\circ\pi_n$, for an appropriate $\widetilde{v}_t^{(n)} : W_n \to W_n$. The use of the notation ∇ and δ below both in (W,μ) and in (W_n,μ_n) should cause no confusion.

Lemma 5.5 *For all* $t \in I$ *and* $n \in \mathbb{N}$, $\widetilde{v}_t^{(n)} \in \mathbb{D}_{1,1}^{W_n}(W_n)$ *and*

$$\nabla \widetilde{v}_t^{(n)} \circ \pi_n = \nabla v^{(n)} = E_n \left(\pi_n (\nabla^H u_t) \pi_n \right) + E \left(\pi_n (\nabla^W B_t) \pi_n \right), \tag{5.12}$$

$$\delta \widetilde{v}_t^{(n)} \circ \pi_n = \delta v^{(n)} = E_n \left(\delta v_t \right) \tag{5.13}$$

Proof of the Lemma: Since (5.13) has already been obtained in Proposition 3.6, we only need to prove (5.12). Indeed, for $N > n$ and $\varphi \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$ denote

$$\widehat{\varphi}^{\,n}(x_1, \ldots, x_n) = \int_{\mathbb{R}^{N-n}} \!\!\cdots\!\! \int \varphi(x_1, \ldots, x_N) d\mu_{N-n}(x_{n+1}, \ldots, x_N). \tag{5.14}$$

If $\Phi = \varphi(\delta e_1, \ldots, \delta e_N)$ then $E_n \Phi = \widehat{\varphi}^{\,n}(\delta e_1, \ldots, \delta e_n) \in \mathcal{S}$, and since clearly $\frac{\partial \widehat{\varphi}^{\,n}}{\partial x_i} = \widehat{\frac{\partial \varphi}{\partial x_i}}^{\,n}$ for each $i = 1, \ldots, n$, it follows that

$$\nabla E_n \Phi = \sum_{i=1}^n \frac{\partial \widehat{\varphi}^{\,n}}{\partial x_i}(\delta e_1, \ldots, \delta e_n) e_i = \sum_{i=1}^N \widehat{\frac{\partial \varphi}{\partial x_i}}^{\,n}(\delta e_1, \ldots, \delta e_n) e_i \pi_n = E_n(\nabla \Phi) \pi_n.$$

For any linear subspace Z of W, separable Banach space Y and $p \in [1, \infty)$, this identity can be extended by linearity and density to

$$\nabla E_n F = E_n \left(\nabla^Z F \right) \pi_n \qquad \forall F \in \mathbb{D}_{p,1}^Z(Y). \tag{5.15}$$

On the other hand it is straightforward to check that

$$\nabla^Z \pi_n F = \pi_n \nabla^Z F \qquad \forall F \in \mathbb{D}_{p,1}^Z(Y) \tag{5.16}$$

and (5.12) is a direct consequence of (5.15) and (5.16). $\square$

Returning to the proof of the theorem, and recalling the definition (5.9),

$$\begin{aligned}
\Gamma^{(\widetilde{v}^{(n)})}(\tfrac{\theta}{2}) &= E \int_a^b \exp \tfrac{\theta}{2} \Big(\|\nabla \widetilde{v}_t^{(n)}\|_{L(W)} + |\delta \widetilde{v}_t^{(n)}| \Big)\, dt \\
&\overset{\text{Lemma 5.5}}{=} E \int_a^b \exp \tfrac{\theta}{2} \Big(\|E_n(\pi_n(\nabla^W v_t)\pi_n\|_{L(W)} + |E_n \delta v_t| \Big)\, dt \\
&\leq E \int_a^b \exp \tfrac{\theta}{2} E_n \Big(\|\pi_n \nabla^H v_t \pi_n\|_{L(H)} + \|\pi_n \nabla^W B_t \pi_n\|_{L(W)} + |\delta v_t| \Big)\, dt
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{Jensen}}{\leq} && E\int_a^b \exp\tfrac{\theta}{2}\Big(\|\nabla^{H} u_t\|_{L(H)} + \|\pi_n \nabla^{W} B_t \pi_n\|_{L(W)} + |\delta u_t|\Big)\,dt \\
&\leq && \left[E\int_a^b \exp\theta\Big(\|\nabla^{H} u_t\|_{L(H)} + |\delta u_t|\Big)\,dt\right]^{\frac{1}{2}} \cdot \\
& && \qquad \cdot \left[\sup_n E\int_a^b \exp\theta\|\pi_n \nabla^{W} B_t \pi_n\|\,dt\right]^{\frac{1}{2}} \\
&\leq && \Big(\Gamma_H(\theta)\Gamma_W(\theta)\Big)^{\frac{1}{2}}. \qquad (5.17)
\end{aligned}
$$

In particular, $\forall t \in I,\ \ \widetilde{v}_t^{(n)} \in \bigcap_{p\geq 1} L^p(\mu_n; W_n) \subset C^\infty(W_n, W_n)$, by the Sobolev embedding theorem. Consequently, recalling the facts presented above in the finite dimensional setup, $\widetilde{v}_t^{(n)}$ generates a quasiinvariant flow $\{\widetilde{T}_{s,t}^{(n)},\ s,t\in I\}$ on W_n satisfying

$$
\widetilde{T}_{s,t}^{(n)}(x) = x + \int_s^t \widetilde{v}_r^{(n)}\left(\widetilde{T}_{s,r}^{(n)}(x)\right) dr\ , \qquad x \in W_n \tag{5.18}
$$

whose Radon–Nikodym derivative

$$
\widetilde{\rho}_{s,t}^{(n)} := \frac{d\left(\widetilde{T}_{s,t}^{(n)*}\mu_n\right)}{d\mu_n} = \exp\int_s^t \left(\delta\widetilde{v}_r^{(n)}\right)\circ \widetilde{T}_{t,r}^{(n)}\,dr \tag{5.19}
$$

satisfies, for any $p>1$ and for all $s,t\in I$ for which $|t-s| < \frac{\theta}{2p}$,

$$
E_n\left(\widetilde{\rho}_{s,t}^{(n)}\right)^p \leq e^{1/p}\left(1 + \tfrac{2(p-1)}{\theta}\sqrt{\Gamma_H(\theta)\Gamma_W(\theta)}\right). \tag{5.20}
$$

Here (5.10) was applied with $\eta = \widetilde{\rho}^{(n)}$ and θ replaced by $\frac{\theta}{2}$, and making use of (5.17).

The flow $\widetilde{T}_{s,t}^{(n)}$ can now be "lifted" from W_n to W by defining

$$
T_{s,t}^{(n)}(\omega) := \widetilde{T}_{s,t}^{(n)}(\pi_n w) + (\omega - \pi_n\omega).
$$

It is straightforward to verify that $T_{s,t}^{(n)}$ satisfies the flow equation

$$
T_{s,t}^{(n)}(\omega) = \omega + \int_s^t v_r^{(n)}\left(T_{s,r}^{(n)}(\omega)\right) dr. \tag{5.21}
$$

and that

$$\rho_{s,t}^{(n)} := \frac{d\left(T_{s,t}^{(n)*}\mu\right)}{d\mu} = \widetilde{\rho}_{s,t}^{(n)} \circ \pi_n = \exp \int_s^t \left(\delta v_r^{(n)}\right) \circ T_{t,r}^{(n)}\, dr \tag{5.22}$$

The solution's construction will be completed by showing convergence of $T_{s,t}^{(n)}$ to a W–valued process $T_{s,t}$ which will be the required flow.

Proposition 5.6 *Let $\eta_t^{[0]}, \eta_t^{[1]}$, $t \in I$, be two C^1 cylindrical time-dependent vector-fields on W such that $\Gamma_i(\theta) := \Gamma^{(\eta^{[i]})}(\theta) < \infty$, for some $\theta > 0$ and $i = 0, 1$ $\big($cf. (5.9) — $\Gamma^{(\eta^{[i]})}(\theta)$ should be understood to mean $\Gamma^{(\widetilde{\eta}^{[i]})}(\theta)$ where $\eta^{[i]} = \widetilde{\eta}^{[i]} \circ \pi_n$ for some $n = n(i)$ $\big)$.*
Let $\{L_{s,t}^{[i]}, s, t \in I\}$ be the unique flow generated on I by $\eta^{[i]}$ as above. Then for any $p > 1$ there exists a finite positive constant $c = c\,(p, \theta, \Gamma_0(\theta), \Gamma_1(\theta))$, increasing in its third and fourth arguments, such that for any $s \in I$

$$E \sup_{\substack{t\in I \\ |t-s| \le \frac{p-1}{2p}\theta}} \left\| L_{s,t}^{[1]} - L_{s,t}^{[0]} \right\|_W \le c \left(\int_I E \left\| \eta_r^{[1]} - \eta_r^{[0]} \right\|_W^p \, dr \right)^{\frac{1}{p}} \tag{5.23}$$

Proof of the Proposition: For a fixed $p > 1$ denote $\Delta_s = \{t \in I, \quad |t - s| \le \frac{p-1}{2p}\theta\}$. Next, let $D_t = \eta_t^{[1]} - \eta_t^{[0]}$ and for every $\lambda \in [0, 1]$ consider the interpolated vector field $\eta_t^{[\lambda]} = \lambda \eta_t^{[1]} + (1-\lambda)\eta_t^{[0]} = \eta_t^{[0]} + \lambda D_t$. Note that by convexity

$$\Gamma^{\eta^{[\lambda]}}(\theta) \le \Gamma_0(\theta) + \Gamma_1(\theta) \tag{5.24}$$

for every $\lambda \in [0, 1]$. Now let $L_{s,t}^{[\lambda]}$ be the flow $\eta^{[\lambda]}$ generates on I, with induced Radon-Nikodym derivative $\rho_{s,t}^{[\lambda]}$, so that setting $Z_{s,t}^{[\lambda]} = d\eta_{st}^{[\lambda]}/d\lambda$ yields

$$L_{s,t}^{[1]} - L_{s,t}^{[0]} = \int_0^1 Z_{s,t}^{[\lambda]}\, d\lambda \qquad\qquad s, t \in I \tag{5.25}$$

and it holds that

$$Z_{s,t}^{[\lambda]}(\omega) = \int_s^t D_r\left(L_{s,t}^{[\lambda]}\omega\right)\, dr + \int_s^t \nabla\eta_r^{[\lambda]}(L_{s,r}^{[\lambda]}\omega) \;\; Z_{s,r}^{[\lambda]}(\omega)\, dr \ ,$$

and thus by Gronwall's lemma, for all $t\in\Delta_s$

$$\|Z^{[\lambda]}_{s,t}(\omega)\|_W \le \int_{\Delta_s} \|D_r(L^{[\lambda]}_{s,r}\omega)\|_W \, dr \quad e^{\int_{\Delta_s} \|\nabla\eta^{[\lambda]}_r(L^{[\lambda]}_{s,r})\|_{L(W)}\, dr} \quad .$$

Inserting this estimate in (5.25), and denoting $p_0 = \frac{p+1}{2}$, $q_0 = \frac{p+1}{p-1}$ $(\frac{1}{p_0}+\frac{1}{q_0}=1)$, we obtain

$$
\begin{aligned}
E \sup_{t\in\Delta_s} \|L^{[1]}_{s,t}(\omega) - L^{[0]}_{s,t}(\omega)\|_W &\le E \sup_{t\in\Delta_s} \int_0^1 \|Z^{[\lambda]}_{s,t}(\omega)\|_W \, d\lambda \\
\le \quad & E\int_0^1 \left(\int_{\Delta_s} \|D_r(L^{[\lambda]}_{s,r}\omega)\|_W \, dr \; e^{\int_{\Delta_s} \|\nabla\eta^{[\lambda]}_r(L^{[\lambda]}_{s,r}\omega)\|_{L(W)}\, dr} \right) d\lambda \\
\overset{\text{Jensen}}{\le} \quad & \int_0^1 \left(E\int_{\Delta_s} \|D_r(L^{[\lambda]}_{s,r}\omega)\|_W \, dr \; \frac{1}{|\Delta_s|} \int_{\Delta_s} e^{\frac{p-1}{2p}\theta \, \|\nabla\eta^{[\lambda]}_r(L^{[\lambda]}_{s,r}\omega)\|_{L(W)}} \, dr \right) d\lambda \\
\le \quad & \int_0^1 E\left(\int_{\Delta_s} \|D_r(L^{[\lambda]}_{s,r}\omega)\|_W^{p_0} \, dr \right)^{\frac{1}{p_0}} E\left(\int_{\Delta_s} e^{q_0 \frac{p-1}{2p}\theta \, \|\nabla\eta^{[\lambda]}_r(L^{[\lambda]}_{s,r}\omega)\|_{L(W)}} \, dr \right)^{\frac{1}{q_0}} d\lambda \\
\le \quad & \int_0^1 \left(E\int_{\Delta_s} \|D_r(\omega)\|_W^{p_0} \, \rho^{[\lambda]}_{s,r}(\omega)\, dr \right)^{\frac{1}{p_0}} \left(E\int_{\Delta_s} e^{\frac{p+1}{2p}\theta \, \|\nabla\eta^{[\lambda]}_r(\omega)\|_{L(W)}} \, \rho^{[\lambda]}_{s,r}(\omega)\, dr \right)^{\frac{1}{q_0}} d\lambda
\end{aligned}
$$

and apply Hölder's inequality twice with the conjugate pair $\frac{p}{p_0}$, $\frac{p}{p-p_0}$:

$$
\begin{aligned}
\le \quad & \int_0^1 \left(E\int_{\Delta_s} \|D_r(\omega)\|_W^{p} \, dr \right)^{\frac{1}{p}} \left(E\int_{\Delta_s} \rho^{[\lambda]}_{s,r}(\omega)^{\frac{p}{p-p_0}} \, dr \right)^{\frac{p-p_0}{pp_0}} \cdot \\
& \cdot \left(E\int_{\Delta_s} e^{\theta \, \|\nabla\eta^{[\lambda]}_r(\omega)\|_{L(W)}} \, dr \right)^{\frac{p_0}{q_0 p}} \left(E\int_{\Delta_s} \rho^{[\lambda]}_{s,r}(\omega)^{\frac{p}{p-p_0}} \, dr \right)^{\frac{p-p_0}{pq_0}} d\lambda \; .
\end{aligned}
$$

The product of the second and fourth factors in the integrand may be estimated using (5.10)

$$
\begin{aligned}
\left(\int_{\Delta_s} E{\rho^{[\lambda]}_{s,r}}^{\frac{2p}{p-1}} \, dr \right)^{\frac{p-1}{2p}} &\le \left(|\Delta_s| e^{\frac{p-1}{2p}} \left(1 + \frac{p+1}{p-1}\, \frac{1}{\theta} \Gamma^{(\eta^{[\lambda]}_r)}(\theta) \right) \right)^{\frac{p-1}{2p}} \\
&\le \left(\frac{1}{p}\, e^{\frac{p-1}{2p}} \left(\theta(p-1) + (p+1)\Gamma^{(\eta^{[\lambda]}_r)}(\theta) \right) \right)^{\frac{p-1}{2p}}
\end{aligned}
$$

while the third factor is bounded above by $\left(\Gamma^{(\eta_r^{[\lambda]})}(\theta)\right)^{\frac{p-1}{2p}}$. This, in conjunction with (5.24), completes the proof of the Proposition. $\square$

We now apply Proposition 5.6 to the cylindrical vector fields $v_t^{(n)}$ and the flows $T_{s,t}^{(n)}$ they generate as in (5.21). Fix an arbitrary $p > 1$. By Corollary 3.7 $v^{(n)}$ is a Cauchy sequence in $L^p(I\times W, \mathrm{leb}\times\mu\,;\, W)$; it thus follows from the proposition, applied with $\frac{\theta}{2}$, that there exists a $\gamma = \gamma(p,\theta) > 0$ and a W–valued process $\{T_{s,t};\ s,t \in I \text{ and } |s-t| \leq \gamma\}$ such that for all $s \in I$, $\lim_{n\to\infty} \sup_{t\in I:|t-s|\leq\gamma} \|T_{s,t} - T_{s,t}^{(n)}\|_W = 0$ in probability. (Note that the constant c in (5.23) depends (monotonically) on the $\Gamma^{[\tilde{v}^{(n)}]}(\frac{\theta}{2})$'s, but these are uniformly bounded - cf. (5.17)). Furthermore, since we have almost sure uniform convergence along a subsequence, $T_{s,t}$ is almost surely continuous in t.

For all $r, s, t \in I$ any two of which are at a distance not larger than γ, the flow property (5.2) obviously holds for $T^{(n)}$ and thus for T as well. This allows us to extend $T_{s,t}$ to all $s, t \in I$: $T_{s,t}(\omega) = T_{s_{m-1},s_m}(T_{s_{m-2},s_{m-1}}(\cdots T_{s_1,s_2}(T_{s_0,s_1}(\omega))\cdots))$ for any sequence $s = s_0, s_1, \ldots, s_{m-1}, s_m = t$ in I such that $|s_i - s_{i-1}| \leq \gamma$, $1 \leq i \leq m$. This extension $\{T_{s,t},\ s,t \in I\}$ is of course independent of the connecting sequence $\{s_i\}$, is a.s. continuous and satisfies the flow property (5.2) as well.

To show that $T_{s,t}$ is quasiinvariant, fix any $p > 1$ and assume first that $|t-s| < \frac{p-1}{2p}\,\theta$. Since $T_{s,t}^{(n)^*}\mu$ and μ are equivalent for every $n \in \mathbb{N}$ we need to verify that the sequence of respective Radon-Nikodym derivatives $\{\rho_{s,t}^{(n)}\}$ is uniformly integrable. However, this follows immediately from (5.20), and moreover by (5.19)

$$\rho_{s,t} := \frac{dT_{s,t}^*\,\mu}{d\mu} = \lim_{n\to\infty} \rho_{s,t}^{(n)} = \exp \int_s^t (\delta v_r) \circ T_{t,r}\, dr$$

which is (5.6) (the last equality follows from (5.22) and Lemma 3.6). In addition, taking the limit as $n \to \infty$ in (5.20) and applying Fatou's lemma, the bound (5.7) is obtained. The resulting quasiinvariance then holds for an arbitrary pair $s, t \in I$ by connecting s and t, if necessary, by a finite sequence $\{s_i\}$ as above.

We now proceed to show that $T_{s,t}$ satisfies (5.1). This will be achieved as the limit in an appropriate sense of (5.21) as $n\to\infty$, the only nontrivial convergence being that of the

integral term. Indeed, again fixing some $p>1$ and assuming $|t-s|<\frac{p-1}{2p}\,\theta$,

$$\begin{aligned}\int_s^t E\left\|v_r(T_{s,r}(\omega)) - v_r^{(n)}(T_{s,r}^{(n)}(\omega))\right\|_W dr \;\; &\leq \;\; \int_s^t E\left\|v_r(T_{s,r}(\omega))-v_r(T_{s,r}^{(n)}(\omega))\right\|_W dr \\ &\quad + \int_s^t E\left\|v_r\left(T_{s,r}^{(n)}(\omega)\right)-v_r^{(n)}\left(T_{s,r}^{(n)}(\omega)\right)\right\|_W dr \end{aligned} \tag{5.26}$$

Let $\varepsilon>0$ and $\frac{1}{p}+\frac{1}{q}=1$. Since

$$\begin{aligned} E\int_s^t &\left(\|v_r(T_{s,r}(\omega))\|_W + \|v_r(T_{s,r}^{(n)}(\omega))\|_W\right) dr \\ &= \int_s^t E\|v_r\|_W \left(\rho_{s,r} + \rho_{s,r}^{(n)}\right) dr \\ &\leq \left(\left(\int_s^t \rho_{s,r}^q dr\right)^{\frac{1}{q}} + \left(\int_s^t \rho_{s,r}^{(n)^q} dr\right)^{\frac{1}{q}}\right) \left(\int_s^t E\|v_r\|_W^p \, dr\right)^{\frac{1}{p}} \\ &\leq M\,\|\mathbf{v}\|_{L^p(\mathrm{leb}\times\mu)} < \infty \end{aligned}$$

where M doesn't depend on n, there exists an $\eta>0$ such that for every measurable $B\subset [s,t]\times W$ with $(\mathrm{leb}\times\mu)(B)\leq\eta$

$$\iint_B \left(\|v_r(T_{s,r}(\omega))\|_W + \|v_r(T_{s,r}^{(n)}(\omega))\|_W\right) dr\, d\mu \leq \varepsilon.$$

In particular choose the set B provided by Lusin's theorem ($(\mathrm{leb}\times\mu)(B)\leq\eta$) and the bounded continuous function g on $[s,t]\times W$ for which $v\mathbf{1}_{B^C}=g\mathbf{1}_{B^C}$ a.e. Then, splitting the expectation over B and B^C,

$$\begin{aligned}\limsup_{n\to\infty} &\int_s^t E\left\|v_r(T_{s,r}(\omega))-v_r(T_{s,r}^{(n)}(\omega))\right\|_W dr \\ &\leq \quad \varepsilon + \lim_{n\to\infty}\int_s^t E\left\|g(T_{s,r}(\omega))-g(T_{s,r}^{(n)}(\omega))\right\|_W dr \;=\; \varepsilon.\end{aligned}$$

Since ε was arbitrary, the first term in the right hand side of (5.26) goes to zero. On the other hand, and referring to (5.20), the second term is bounded by

$$\int_s^t E\|v_r-v_r^{(n)}\|_W \rho_{s,r}^{(n)}\, dr \leq C\left(\int_s^t E\|v_r-v_r^{(n)}\|_W^p\, dr\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0$$

with $C=\Big((t-s)e^{1/q}\Big(1+\frac{2(q-1)}{\theta}\sqrt{\Gamma_H(\theta)\Gamma_W(\theta)}\Big)\Big)^{\frac{1}{q}}$. Thus for all $|t-s|<\frac{p-1}{2p}\,\theta$ all the terms of (5.21) converge to those of (5.1). The latter equation then holds as well for any arbitrary pair $s,t\in I$ as a result of T's flow property which has already been proved.

Finally to show uniqueness we first observe that since the vector field v_t was assumed to possess continuous paths almost surely, any flow $S_{s,t}$ generated by it is a.s. continuously differentiable both in t and in s. For such a flow define, for any fixed $s\in I$,

$$U_{s,t}=T_{t,s}\circ S_{s,t}, \qquad\qquad t\in I\,,$$

where T is the particular flow constructed above. Our aim is to show that $\dot U_{s,t}=0$ for all $t\in I$, a.s. (here $\dot A_t:=\frac{d}{dt}A_t$). Indeed

$$\dot U_{s,t}=\dot T_{t,s}(S_{t,s})+\nabla^W T_{s,t}(S_{s,t})\,v_t(S_{s,t})=\Big(\dot T_{t,s}+\nabla^W T_{s,t}\,v_t\Big)\circ S_{s,t}. \tag{5.27}$$

For every fixed t, and by quasiinvariace, this expression will be 0 for one flow $S_{s,t}$ if it is so for any other. Since it *is* zero when $S_{s,t}=T_{s,t}$ (in which case $U_{s,t}(\omega)=\omega$), we conclude that $\dot U_{s,t}=0$ a.s., $\forall t\in I$, for *any* flow $S_{s,t}$. The quantifiers a.s. and $\forall t$ can now be interchanged since the processes in (5.27) have a.s. continuous paths. This completes the proof of Theorem 5.3. $\square$

The next result shows that the existence of $\mathbf{v}$'s divergence in Theorem 5.3 is "nearly" a necessary assumption.

Theorem 5.7 *Let* $\{v_t\}_{t\in I}:W\to W$ *be a time dependent vector field which generates a quasiinvariant flow* $T_{s,t}$ *with Radon-Nikodym derivative* $\Lambda_{s,t}$. *Then* $v_s\in\mathrm{dom}_1\delta$ *for any* $s\in I$ *for which*

$$\Lambda'_{s,s}:=\lim_{t\to s}\frac{\Lambda_{s,t}-1}{t-s} \tag{5.28}$$

exists weakly in $L^1(\mu)$, *in which case* $\delta v_s=\Lambda'_{s,s}$. *In particular, if* $T_{s,t}$ *is measure preserving on* I *then* δv_s *exists and is zero for every* $s\in I$.

Proof: Let $\Phi\in\mathcal{S}$ be arbitrary. It is a direct consequence of (5.1) that for every $s,t\in(a,b)$

$$\Phi(X_{s,t})=\Phi(X_{s,s})+\int_s^t {}_W\langle v_r(X_{s,r}),\nabla\Phi(X_{s,r})\rangle_{W^*}\,dr \qquad \text{a.s.}$$

and thus

$$
\begin{aligned}
E\Lambda_{s,t}\Phi = E\Phi(X_{s,t}) &= E\Phi + \int_s^t E\,{}_W\langle v_r \circ X_{s,r}, (\nabla\Phi)\circ X_{s,r}\rangle_{W^*}\, dr \\
&= E\Phi + \int_s^t E\Lambda_{s,r}\,{}_W\langle v_r, \nabla\Phi\rangle_{W^*}\, dr.
\end{aligned}
$$

It follows that $E\Lambda_{s,t}\Phi$ is differentiable in t and that

$$
E\Lambda'_{s,s}\Phi = \frac{\partial\{E\Lambda_{s,t}\Phi\}}{\partial t}\Big|_{t=s} = E\,{}_W\langle v_s, \nabla\Phi\rangle_{W^*} = E(\Phi\delta v_s)\,,
$$

where the first equality results from the existence of (5.28), thus proving the theorem. □

6 Additional Remarks

In this final section we address two additional aspects of the flows introduced in Section 5: their adaptedness and their relevance to an associated PDE.

I. Adapted Flows:

Let (W, H, μ) be an AWS and $\{\Pi^\theta, 0 \le \theta \le 1\}$ a continuous, strictly increasing resolution of the identity on H. Let ([20], 2.6 of [21])

$$
\mathcal{F}_\theta = \sigma\Big\{\delta(\Pi^\theta h), h \in H\Big\}, \quad 0 \le \theta \le 1
$$

be the filtration induced by $\Pi^{\cdot}$ on (W, H, μ). In what follows we assume that $\Pi^\theta W^* \subset W^*$ for all $\theta \in [0,1]$. This can be easily verified for the classical Wiener space.

Definition 6.1 ([20] or Section 2.6 of [21]) *An H-valued random variable u is $\mathcal{F}_\theta$ measurable if $(u,h)_H$ is $\mathcal{F}_\theta$ measurable for all $h \in H$. Moreover, $u \in H$ is* adapted *if $(u, \Pi^\theta h)_H$ is $\mathcal{F}_\theta$ measurable for all $\theta \in [0,1]$.*

Definition 6.2 *A W-valued random variable u is adapted if there exists a sequence of H-valued r.v.'s u_n $n = 1, 2, \ldots$ which are adapted and $|u - u_n|_W \xrightarrow{\text{a.s.}} 0$. Let u be a W or W^{**} valued r.v. then $\Pi^\theta u \in W^{**}$ is defined by*

$$
{}_{W^{**}}\langle \Pi^\theta u, h\rangle_{W^*} = {}_W\langle u, \Pi^\theta h\rangle_{W^*} \qquad \forall h \in W^*\,.
$$

We prepare the following lemma for later reference.

Lemma 6.3 *(a) If U is a quasiinvariant adapted W–valued random variable and α is $\mathcal{F}_\theta$–measurable, then $\alpha \circ U$ is also $\mathcal{F}_\theta$–measurable.*
(b) Let U_1, U_2 be as in (a) such that $\Pi^\theta U_1 = \Pi^\theta U_2$ for some $\theta \in [0,1]$. Then if α is a $\mathcal{F}_\theta$–measurable random variable, $a \circ U_1 = \alpha \circ U_2$. Similarly, if v is a W-valued adapted random variable, then $\Pi^\theta(v \circ U_1) = \Pi^\theta(v \circ U_1)$.

Proof: Since α is $\mathcal{F}_\theta$–measurable it is the a.s. limit of polynomials in $\delta(\Pi^\theta h)$, $h \in W^*$ (recall that we have assumed $\Pi^\theta h \in W^*$), so that $\alpha \circ U$ is the a.s. limit of polynomials in $\delta(\Pi^\theta h) \circ U = {}_W\langle U, \Pi^\theta h\rangle_{W^*}$. Since U is adapted, each ${}_W\langle U, \Pi^\theta h\rangle_{W^*}$ is $\mathcal{F}_\theta$–measurable and thus so is $\alpha \circ U$. This proves (a), and (b) follows directly from (a). $\square$

Proposition 6.4 *Assume that $v_t(\omega)$ is adapted for every $t \in I$ and satisfies the conditions of Theorem 5.3 (including the a.s. continuity in t). Let $v_t^\theta(\omega) := \Pi^\theta v_t(\omega)$ and assume that $v_t^\theta(\omega)$ satisfies these conditions as well for all $\theta \in [0,1]$. Then the solution to (5.1):*

$$T_{s,t}(\omega) = \omega + \int_s^t v_r(T_{s,r})\, dr$$

is also adapted.

Proof: Let $\widetilde{T}^\theta_{s,t}(\omega)$ solve the equation

$$\widetilde{T}^\theta_{s,t}(\omega) = \omega + \int_s^t v_r^\theta \circ \widetilde{T}^\theta_{s,r}(\omega)\, dr \tag{6.1}$$

hence

$$\Pi^\theta \widetilde{T}^\theta_{s,t}(\omega) = \Pi^\theta \omega + \int_s^t v_r^\theta \circ \widetilde{T}^\theta_{s,r}(\omega)\, dr$$

and $\Pi^\theta \widetilde{T}^\theta_{s,t}(\omega)$ is $\mathcal{F}_\theta$-measurable since it is a measurable function of $\{v_r^\theta, r \in [s,t]\}$ and $\Pi^\theta \omega$.

On the other hand, by (5.1)

$$\Pi^\theta T_{s,t}(\omega) = \Pi^\theta \omega + \int_s^t \Pi^\theta \Big(v_r \circ T_{s,r}(\omega) \Big)\, dr\,.$$

Set $T^\theta_{s,t}(\omega)=\Pi^\theta T_{s,t}(\omega)+(I-\Pi^\theta)\omega$, then

$$T^\theta_{s,t}\omega=\omega+\int_s^t(\Pi^\theta v_r)\circ T_{s,r}(\omega)\,dr=\omega+\int_s^t(\Pi^\theta v_r)\circ T^\theta_{s,r}(\omega)\,dr\,. \tag{6.2}$$

Comparing (6.1) with (6.2) yields, by uniqueness, $\widetilde{T}^\theta_{s,t}(\omega)=T^\theta_{s,t}(\omega)$. Hence $\langle T^\theta_{s,t}(\omega),\Pi^\theta h\rangle=\langle T_{s,t}(\omega),\Pi^\theta h\rangle$ is $\mathcal{F}_\theta$-measurable and since $\theta\in[0,1]$ was arbitrary, $T_{s,t}(\omega)$ is adapted. $\square$

Remark 6.5 The flow considered by Cipriano and Cruzeiro [2] is of the type considered in this proposition. Let (W,H,μ) be the classical d-dimensional Wiener process

$$\omega=\left\{\begin{pmatrix}\omega'(t)\\ \vdots\\ \omega^d(t)\end{pmatrix},\quad t\geq 0\right\}\quad\text{and}\quad\Big(v(\omega)\Big)_.=\begin{pmatrix}(v'(\omega))_.\\ \vdots\\ (v^d(\omega))_.\end{pmatrix}_. \qquad .$$

In the case considered in [2]

$$\Big(v^i(\omega)\Big)_.=\sum_{j=1}^d\int_0^. a_i^j(\omega,s)d\omega^j(s)+\int_0^. b^i(\omega,s)\,ds$$

and for every $1\leq i,j\leq d,\ \ s\geq 0$, the coefficients $a^i_j(\omega,s)$ and $b^i(\omega,s)$ are $\mathcal{F}_s$ measurable. The assumptions in [2] guarantee that v satisfies those of Proposition 6.4.

II. The equation $\frac{df(t,\omega)}{dt}=\delta(\mathbf{A}(\omega)\cdot\nabla f(t,\omega)),\ (\mathbf{A}+\mathbf{A}^T=0)$.

Let $T_t\omega$, $t\in R$, $T_0\omega=\omega$, be a measure preserving transformation on W. $T_t\omega$ is said to be a *stationary process* if for any n, any $t_1,\dots,t_n$, any smooth $\varphi_i(\omega)$ and any τ

$$\text{Law}\,\Big\{\varphi_1(T_{t_1}\omega),\dots,\varphi_n(T_{t_n}\omega)\Big\}=\text{Law}\,\Big\{\varphi_i(T_{t_1+\tau}\omega),\dots,\varphi_n(T_{t_n+\tau}\omega)\Big\} \tag{6.3}$$

A flow which is also a stationary process will be called a *stationary flow*. Note that if $T_t\omega, t\in R$ is a measure preserving flow then it is also a stationary flow.

Proposition 6.6 *Let $\mathbf{A}(\omega)$ be a measurable and skew symmetric transformation on H. Further assume that $\mathbf{A}(\omega)$ transforms $\mathbb{D}_{p,1}(H)$ into $\mathbb{D}_{p,1}(H)$. Let $B(\omega)=\sum_1^\infty\delta(\mathbf{A}(\omega)e_i)e_i$, where $e_i, i=1,2,\dots$ is a smooth ONB on H, converges in $L^1(\mu;W)$, and assume that $T_t\omega$, $t\in R$ solves*

$$\frac{dT_t\omega}{dt}=B(T_t\omega),\qquad T_0\omega=\omega$$

and $T_t\omega$, $t \in R$, is a stationary process. Let $f_0(\omega)$ be a smooth functional on W, for which $f_0(T_t\omega) \in \mathbb{D}_{p,2}$.

Then $f(t,\omega) := f_0(T_t\omega)$ solves the equation:

$$\frac{df(t,\omega)}{dt} = \delta(\mathbf{A}(\omega)\nabla f(t,\omega)), \tag{6.4}$$

$f(0,\omega) = f_0(\omega)$.

Proof: For any smooth $\varphi(\omega)$ we have by stationarity

$$\begin{aligned} E\frac{1}{\varepsilon}\Big[\Big(f_0(T_{t+\varepsilon}\omega) - f_0(T_t\omega)\Big)\varphi(\omega)\Big] &= \frac{1}{\varepsilon}E\Big(f_0(T_t\omega)\cdot\varphi(T_{-\varepsilon}\omega) - f_0(T_t\omega)\varphi(\omega)\Big) \\ &= Ef_0(T_t\omega)\frac{1}{\varepsilon}\Big(\varphi(T_{-\varepsilon}\omega) - \varphi(\omega)\Big) \end{aligned}$$

and since $(d\varphi(T_t\omega)/dt)_{t=0} = \delta(\mathbf{A}(\omega\nabla\varphi(\omega))$ (cf. [9] or eqn. 1.10 of [22])

$$E\varphi(\omega)\frac{df_0(T_t\omega)}{dt} = -E\Big(f_0(T_t\omega)\delta(\mathbf{A}(\omega)\nabla\varphi(\omega))\Big)$$

integrating by parts yields

$$\begin{aligned} E\varphi(\omega)\frac{df_0(T_t\omega)}{dt} &= -E\Big(\nabla(f_0(T_t\omega)), \mathbf{A}\nabla\varphi\Big) \\ &= E\Big(\delta(\mathbf{A}\nabla f_0(T_t\omega))\varphi(\omega)\Big) \end{aligned}$$

and (6.4) follows. □

Corollary 6.7 *If in addition to the assumptions of proposition 6.6, $\nabla f(t,\omega) \in W^*$ a.s. for every t, then*

$$\frac{df(t,\omega)}{dt} = {}_W\langle B(\omega), \nabla f(t,\omega)\rangle_{W^*} \qquad f(0,\omega) = f_0(\omega) \tag{6.5}$$

Proof: Let $\nabla f(t,\omega) = \sum_i \psi_i(t,\omega) e_i$, then since $f_0(t,\omega) \in \mathbb{D}_{p,2}$, $\psi_i(t,\omega) \in \mathbb{D}_{p,1}$ and

$$
\begin{aligned}
\delta(\mathbf{A}\nabla f) &= \sum_1^\infty \delta\Big(\psi_i(t,\omega)\mathbf{A}(\omega)e_i\Big) \\
&= \sum_1^\infty \psi_i(t,\omega)\delta\Big(\mathbf{A}(\omega)e_i\Big) - \sum_1^\infty \nabla\psi_i(t,\omega), \mathbf{A}(\omega)e_i \\
&= \langle B(\omega), \nabla f(t,\omega)\rangle - \sum_i \sum_j \nabla^2_{e_i,e_j} f(t,\omega)(e_j, \mathbf{A}e_i) \\
&= \langle B(\omega), \nabla f(t,\omega)\rangle - 0\,.
\end{aligned}
$$

$\square$

In the converse direction

Proposition 6.8 *Let $\mathbf{A}(\omega)$ be a measurable and skew symmetric transformation on H transforming $\mathbb{D}_{2,1}(H)$ into $\mathbb{D}_{2,1}(H)$. Assume that $f^j(t,\omega) \in \mathbb{D}_{p,2}$ and $\nabla f^j \in W^*$, $j = 1,2,\ldots$ solves equation (6.4) for all t and for the initial condition $f^j(0,\omega) = {}_W\langle\omega, e_j\rangle_{W^*}$ where $e_j, j = 1,2,\ldots$ is a smooth ONB on H. Then*

(a) $\sum_1^n \beta_j f^j(t+\tau_j,\omega)$ *solves* (6.4) *under the initial condition* $\sum_1^n \beta_j f^j(\tau_j,\omega)$ *for any* β_j, τ_j.

(b) Let $\psi^n(t,\omega) = \exp i \sum_1^n f^j(t+\tau_j,\omega)$, *then* $\psi^n(t,\omega)$ *solves* (6.4) *under the initial condition*

$$\psi^n(0,\omega) = \exp i \sum_1^n f^j(\tau_j,\omega)\,.$$

(c) $T_t\omega$ *defined by*

$$T_t\omega = \sum_{j=1}^\infty f^j(t,\omega)e_j\,,$$

is a stationary process.

(d) If moreover,(6.4) *possesses a unique solution for every initial* $f_0(\omega) \in \mathbb{D}_{p,2}$, *then* $T_s(T_t\omega) = T_{s+t}\omega$.

Proof: (a) is trivial. (b) since $f^j(t,\omega) \in \mathbb{D}_{p,2}$ and $\mathbf{A} + \mathbf{A}^T = 0$

$$\begin{aligned}
\delta\Big(\mathbf{A}\nabla f^k(t,\omega)\Big) &= \sum_j \delta(\nabla_{e_j} f^k(t,\omega)\mathbf{A}e_j) \\
&= \sum_j \nabla_{e_j} f^k(t,\omega)\delta(\mathbf{A}e_j) - \sum_j \sum_k \nabla^2_{e_i,e_j} f^k(t,\omega)(e_k,\mathbf{A}e_j) \\
&= {}_W\langle \delta\mathbf{A}, \nabla f^k(t,\omega)\rangle_{W^*}\,. \qquad (6.6)
\end{aligned}$$

On the other hand, differentiating $\psi^n(t,\omega)$ with respect to t and applying (6.6) yields

$$\begin{aligned}
\frac{d\psi^n(t,\omega)}{dt} &= \psi^n(t,\omega)\,{}_W\langle \delta\mathbf{A}, \sum_{k=1}^n \nabla f^k(t+\tau_j,\omega)\rangle_{W^*} \\
&= {}_W\langle \delta\mathbf{A}, \nabla\psi^n(t,\omega)\rangle_{W^*} \\
&= \delta\Big(\mathbf{A}\nabla\psi^n(t,\omega)\Big)
\end{aligned}$$

proving (b). Turning to (c), by the last equation for $\tau_j = 0$, $j = 1, 2, \ldots,$

$$E\psi^n(t,\omega) = E\psi^n(0,\omega) \qquad (6.7)$$

and since e_i is an ONB, it follows that $f^i(t,\omega)$ are $N(0,1)$ i.i.d. random variables; hence by the Ito-Nisio theorem $T_t\omega$ is a measure invariant transformation. Moreover, $T_t\omega$ is a stationary process since (6.7) holds for any t and any $\tau_1, \tau_2, \ldots$.

To show (d) note first that $f^i(t,\omega) = \langle T_t\omega, e_i\rangle$ solves (6.4) under $f^i(0,\omega) = \langle\omega, e_i\rangle$, hence $\langle T_{t+\tau}\omega, e_i\rangle$ solves the same equation under $f^i_0(\omega) = \langle T_\tau\omega, e_i\rangle$. On the other hand, as in (b), for any smooth $f_0(\omega) = \widetilde{f}_0(\langle\omega, e_1\rangle, \ldots \langle\omega, e_n\rangle)$ $f_0(T_t\omega) = \widetilde{f}_0(\langle T_t\omega, e_1\rangle, \ldots \langle T_t\omega, e_n\rangle)$ solves (6.4). In particular set $f^i_0(\omega) = \langle T_\tau\omega, e_i\rangle$, then

$$\begin{aligned}
f^i(t,\omega) &= \langle T_\tau\omega, e_i\rangle \circ T_t\omega \\
&= \langle T_\tau(T_t\omega)\rangle, e_i\rangle
\end{aligned}$$

and (d) follows by the uniqueness of the solution to (6.4). □

References

[1] V. Bogachev and E. Mayer-Wolf, Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions, *J. Func. Anal.* **167** (1999), 1–68.

[2] F. Cipriano and A.B. Cruzeiro, Flows associated to tangent processes on Wiener space, *J. Funct. Anal.* **166** (1999), 310–331.

[3] A.B. Cruzeiro, Équations differentielles sur l'espace de Wiener et formules de Cameron–Martin non linéaires, *J. Funct. Anal.* **54** (1983), 206–227.

[4] A.B. Cruzeiro and P. Malliavin, Renormalized differential geometry on path space, structural equation, curvature, *J. Funct. Anal.* **139** (1996), 119–181.

[5] A.B. Cruzeiro and P. Malliavin, A class of anticipative tangent processes on the Wiener space, *C.R. Acad. Sci. Paris,* **333(1)** (2001), 353–358.

[6] B.K. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact manifold, *J. Funct. Anal.* **110** (1992), 272–376.

[7] B.K. Driver, Towards calculus and geometry on path spaces, *Symp. Proc. Pure Math.* **57** (1995), 405–422.

[8] D. Feyel and de La Pradelle, Espaces de Sobolev gaussiens, *Ann. Inst. Fourier* **41**, (1991), 49–76.

[9] Y. Hu, A.S. Üstünel and M. Zakai, Tangent processes on Wiener space, *J. Funct. Anal.* **192** (2002), 234–270.

[10] S. Kusuoka, Nonlinear transformations containing rotation and Gaussian measure. *J. Math, Sci. Univ. Tokyo,* **10**, (2003), 1–40.

[11] P-L. Lions, Sur les équations différentielles ordinaires et les équations de transport, *C.R. Acad. Sci Paris* **326(1)**, (1998) 833–838.

[12] M. Ledoux and M. Talagland, *Probability in Banach spaces*, Springer 1991.

[13] P. Malliavin, *Stochastic Analysis*, Springer-Verlag, Berlin/New York, 1997.

[14] P. Malliavin and D. Nualart, Quasi sure analysis of stochastic flows and Banach space valued smooth functionals on the Wiener space, *J. Funct. Anal.* **112** (1993), 287–317.

[15] D. Nualart and M. Zakai, A summary of some identities of the Malliavin calculus, In Stochastics Partial Differential Equations and Applications II, G. Da Prato and L. Tubero, editors. *Lecture Notes in Mathematics* **1390**, 192–196, Springer 1989.

[16] G. Peters, Anticipating flows on the Wiener space generated by vector fields of low regularity, *J. Funct. Anal.* **142** (1996) 129–192.

[17] I. Shigekawa, De Rham-Hodge-Kodaira's decomposition on an abstract Wiener space, *J. Math. Kyoto Univ.* **26**, 191–202 (1986).

[18] I. Shigekawa, Sobolev spaces of Banach-valued functions associated with a Markov process, *Prob. Th. Related Fields,* **99** (1994) 425–441.

[19] A.S. Üstünel, An Introduction to Analysis of Wiener Space, *Lecture Notes in Math.* **1610**, Springer 1996.

[20] A.S. Üstünel and M. Zakai, The construction of filtrations on abstract Wiener space, *J. Funct. Anal.* **143** (1997) 10–32.

[21] A.S. Üstünel and M. Zakai, *Transformation of Measure on Wiener Space*, Springer-Verlag, New York/Berlin, 1999.

[22] M. Zakai, Rotations and tangent processes on Wiener space, *Seminaire de Probabilities* XXXVIII 2004 to appear. (arXiv:math. PR/0301351).